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Research article

Topp-Leone Weibull Generated Family of Distributions with Applications

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Abstract: We reveal and investigate the Topp-Leone (TL) Weibull G (TLWG) family, a new generator of continuous lifespan distributions, as a consequence of this article. However, various of its statistical characteristics have been proposed. Examine the TLWG family, a new source of continuous lifespan distributions. The parameters of maximum likelihood (MLL) estimations are predicted. We can see the significance and versatility of the recommended family of algorithms while using the TLW exponential model as an example of the new recommendation family with the help of two real-world examples.

Keywords: Topp-Leone Family; Weibull Family, Maximum Likelihood Estimation.

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1. Introduction

In the past few years, there has been a strong emphasis on developing more adjustable distributions. During the last few decades, a wide range of G families of distributions have been designed and built and explored to simulate real-life data from various practical fields of study such as finance, environmental studies, engineering, biological research, medical sciences, and insurance. We created a variety of distributions by making generalizations G families. At least one shape parameter is combined with the baseline parameter in these new families, allowing for greater adaptability. For instance, the generalized transmuted exponentiated G by Yousof et al. (2015), Weibull (W) G (WG) by Bourguignon et al. (2014), the Burr type X-G by Yousof et al. (2017a), Type II half logistic G by Hassan et al. (2017), exponentiated transmuted G by Merovci et al. (2017), truncated Cauchy power G by Aldahlan et al. (2020), transmuted odd Fréchet G by Badr et al. (2020), the beta W G by Yousof et al. (2017b), odd generalized N-H by Ahmad et al. (2020), weighted Shanker distribution by Helal et al. (2022), the generalized odd W G by Korkmaz et al. (2018), exponentiated M-G by Bantan et al. (2020a), the transmuted W G by Alizadeh et al. (2018), A new W G by Yousof et al. (2018a), TL G by Rezaei et al. (2017), X-Gamma Lomax distribution by Almetwally et al. (2022), Type 2 TL G by Elgarhy et al. (2018), sine TL G by Al-Babtain et al. (2020), TL odd Fréchet G by Al-Marzouki (2020), Type 2 power TL G by Bantan et al. (2020b), a new version of Power TL G by Bantan et al. (2019) and Type 2 generalized TL G by Hassan et al. (2019), among others.

Rezaei et al. (2017) proposed the cumulative distribution function (CDF) and the density function (PDF) of the TL G (TLG) as below

$$F_{\alpha}(z) = \left[1 - \overline{G}^{2}_{\underline{\Phi}}(z)\right]^{\alpha} = G^{\alpha}_{\underline{\Phi}}(z)\left[2 - G_{\underline{\Phi}}(z)\right]^{\alpha}, \tag{1.1}$$

and

$$f_{\alpha}(z) = 2\alpha g_{\underline{\Phi}}(z)\left[1 - \overline{G}^{2}_{\underline{\Phi}}(z)\right]^{\alpha-1}\overline{G}_{\underline{\Phi}}(z). \tag{1.2}$$

Bourguignon et al. (2014) investigated the WG family's CDF as below

$$G_{\underline{\Phi}}(z) = G_{\beta}(z) = 1 - exp\left[-O_{\underline{\Phi}}(z)^{\beta}\right], \tag{1.3}$$

where, $O_{\underline{\Phi}}(z) = \frac{G_{\underline{\Phi}}(z)}{\overline{G}_{\underline{\Phi}}(z)}$. Then utilizing (1.3) and (1.1), the CDF of the TL W G (TLWG) is provided via

$$F_{\alpha,\beta,\underline{\Phi}}(z) = \left\{1 - exp\left[-2O_{\underline{\Phi}}(z)^{\beta}\right]\right\}^{\alpha}, \tag{1.4}$$

the corresponding PDF is

$$f_{\alpha,\beta,\underline{\Phi}}(z) = 2\alpha\beta g_{\underline{\Phi}}(z)\, exp\left[-2O_{\underline{\Phi}}(z)^{\beta}\right]\frac{G_{\underline{\Phi}}(z)^{\beta-1}}{\overline{G}_{\underline{\Phi}}(z)^{\beta+1}}\left\{1 - exp\left[-2O_{\underline{\Phi}}(z)^{\beta}\right]\right\}^{\alpha-1}. \tag{1.5}$$

The hazard rate function (HRF) is simply determined utilizin $h_{\alpha,\beta,\underline{\Phi}}(z) = f_{\alpha,\beta,\underline{\Phi}}(z)/\left[1 - F_{\alpha,\beta,\underline{\Phi}}(z)\right]$.

The remainder of the paper can be structured as follows: Section 2 begins by introducing some important new family expansions; Section 3 describes the distribution's features. Section 4 discusses some new sub-models of the proposed family. Section 5 presents the MLL method of parameter estimation for the recommended family. Section 6 examines the applications of two real-world data sets. Finally, in section 7, we discuss the current study's findings and conclusions.

2. Important Representation

Examine the binomial theory supplied as

$$\left(1 - \frac{a_1}{a_2}\right)^{a_3 - 1} = \sum_{l=0}^{\infty} (-1)^l \binom{a_3 - 1}{l} \left(\frac{a_1}{a_2}\right)^l |_{\left(b>0 \text{ and } |\frac{a_1}{a_2}|<1\right)}, \tag{2.1}$$

The PDF in (1.5) may then be demonstrated as

$$f_{\alpha,\beta,\underline{\Phi}}(z) = 2\alpha\beta g_{\underline{\Phi}}(z) \frac{G_{\underline{\Phi}}(z)^{\beta-1}}{\overline{G}_{\underline{\Phi}}(z)^{1+\beta}} \sum_{l=0}^{\infty} (-1)^l \binom{\alpha - 1}{l} \underbrace{exp\left[-2(1+l)O_{\underline{\Phi}}(z)^{\beta}\right]}_{A(z;\beta,\underline{\Phi})}, \tag{2.2}$$

Through using power series expansion on A(z), we get

$$A(z;\beta,\underline{\Phi}) = \sum_{l=0}^{\infty} \frac{1}{l!} [-2(1+l)]^l O_{\underline{\Phi}}(z)^{\beta l},$$

where $O_{\underline{\Phi}}(z)^{\beta d} = \frac{G_{\underline{\Phi}}(z)^{\beta d}}{\overline{G}_{\underline{\Phi}}(z)^{\beta d}}$. Then

$$f_{\alpha,\beta,\underline{\Phi}}(z) = 2\alpha\beta \sum_{l,d=0}^{\infty} \frac{1}{d!} (-1)^{l+d} [2(1+l)]^d \binom{\alpha - 1}{l} g_{\underline{\Phi}}(z) \frac{G_{\underline{\Phi}}(z)^{(1+d)\beta-1}}{\overline{G}_{\underline{\Phi}}(z)^{(1+d)\beta+1}},$$

but $\overline{G}_{\underline{\Phi}}(z)^{-(1+d)\beta-1} = \sum_{k=0}^{\infty} \binom{\beta(d+1)+1}{k} G_{\underline{\Phi}}(z)^k$. Then, the $f_{\alpha,\beta,\underline{\Phi}}(z)$ can indeed be expressed as

$$f_{\alpha,\beta,\underline{\Phi}}(z) = \sum_{l,d,k=0}^{\infty} v_{(l,d,k)}\ h_{\beta^*}(z)|_{(\beta^*=\beta(1+d)+k)}, \tag{2.3}$$

where

$$v_{(l,d,k)} = 2\alpha\beta \sum_{l,d,k=0}^{\infty} (-1)^{l+d} \binom{\alpha - 1}{l} \binom{\beta(d+1)+1}{k} \frac{[2(l+1)]^d}{d!\,\beta^*},$$

and $h_{\beta^*}(z) = \beta^* g_{\underline{\Phi}}(z) G_{\underline{\Phi}}(z)^{\beta^*-1}$ depict the PDF of the exponentiated G (ExG) distribution using parameter β^*.

3. Statistical Features of The New Family

3.1 Quantile Function

The TLWG quantile function is provided via

$$z = Q(u) = G^{-1}\left\{\frac{\left[-\frac{1}{2}log\left(1-u^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}}{1+\left[-\frac{1}{2}log\left(1-u^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}}\right\}.$$

By deciding to take u as a uniform RVr in (0,1), we could indeed start generating z.

3.2 Moments

The r^{th} moment of TLWG could be obtained in the following manner:

$$\mu_r^{/} = \int_0^\infty z^r\, f(z)dz = \sum_{l,d,k=0}^{\infty} v_{(l,d,k)}\; I_{(0,\infty)}(\beta^*) \tag{3.1}$$

where $I_{(0,\infty)}(\beta^*) = \int_0^\infty z^r\, h_{\beta^*}(z)dz$ is the r^{th} moment of the ExG model utilizing parameter β^*.

4. Some Sub-Models

This section will examine numerous different TLWG relatives situations. We present six special models from the TLWG family that are equivalent to the baseline exponential (Exp), Weibull (We), Lomax (Lo), Burr-X (BX), log-logistic (LoL), and Lindley (Li) distributions. Table 1 includes a listing of the new models.

Table 1. The new models

Base line	New model	CDF
Exp	TLWExp	$\{1-exp[-2(exp(\theta z)-1)^{\beta}]\}^{\alpha}$
We	TLWWe	$[1-exp(-2\{exp[(\theta z)^{b}]-1\}^{\beta})]^{\alpha}$
Lo	TLWLo	$\left(1-exp\left\{-2\left[\left[1+\left(\frac{z}{b}\right)\right]^{\theta}-1\right]^{\beta}\right\}\right)^{\alpha}$
BX	TLWBX	$\left\{1-exp\left[-2\left(\left\{1-exp\left[-(\frac{z}{\theta})^{2}\right]\right\}^{-b}-1\right)^{-\beta}\right]\right\}^{\alpha}$
LoL	TLWLoL	$\left\{1-exp\left[-2\left(\frac{z}{\theta}\right)^{b\beta}\right]\right\}^{\alpha}$
Li	TLWLi	$\left(1-exp\left\{-2\left[\left(\frac{1+\theta+\theta z}{1+\theta}\right)^{-1}-1\right]^{\beta}\right\}\right)^{\alpha}$

The CDF parameters listed above are all positive actual numbers. Figures 1 and 2 display graphs of the TLWExp model's PDF and HRF. Figure 1 indicates that the new PDF can take a variety of useful forms. Figure 2 depicts the new HRF as raising, bathtub, J shape, and declining.

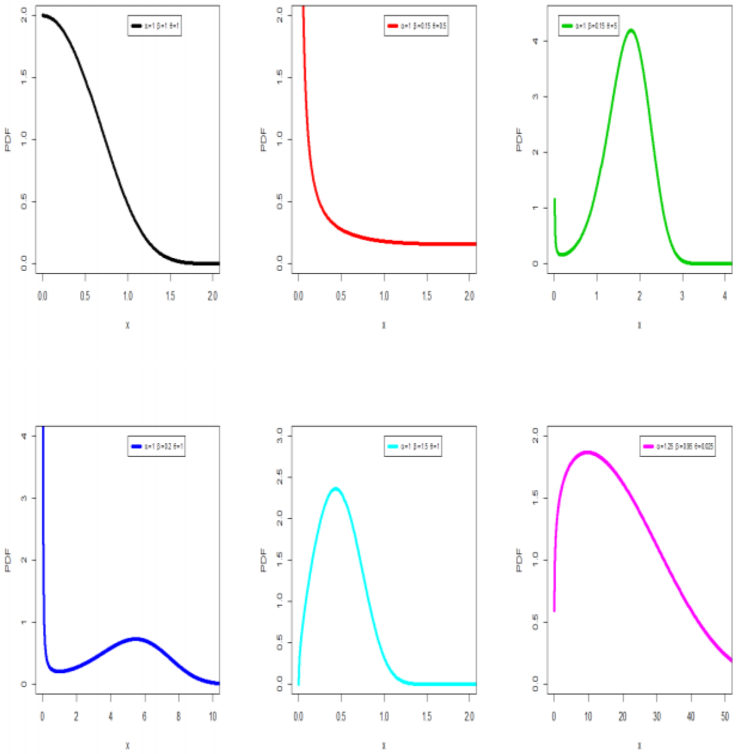

Figure 1. The PDF plots of the TLWExp model.

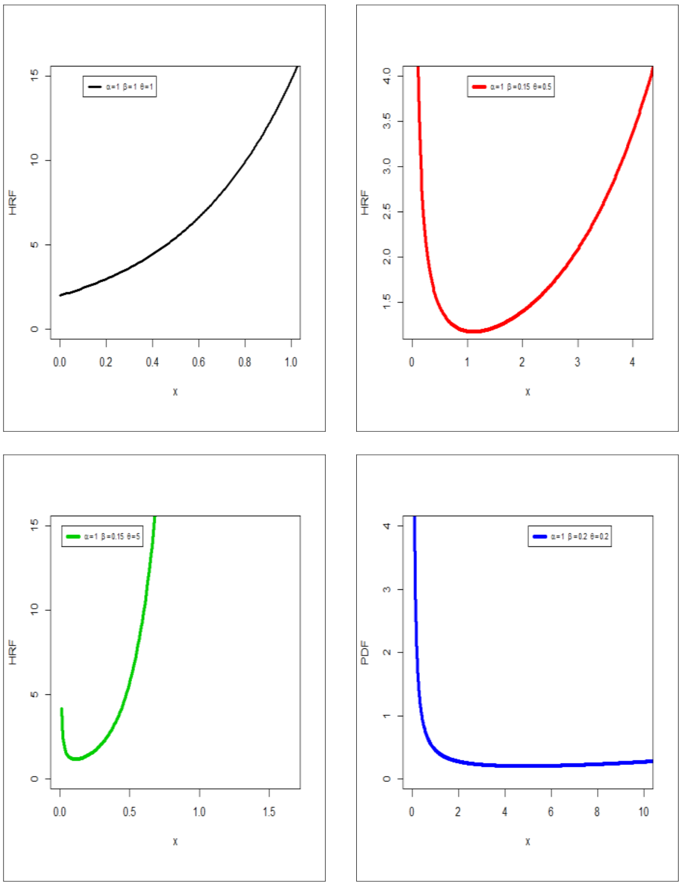

Figure 2. The HRF plots of the TLWExp model.

5. Maximum Likelihood (ML) Approach

Assume that $z_1, \ldots, z_n$ be an n-th random sample from the TLWG generated family of distributions, which has PDF (5). Suppose that $\underline{P} = (\alpha, \beta, \underline{\delta})^T$ is the vector of parameters. The total log-likelihood (TLL) function for $\underline{P}$ is provided as below

$$L_n(\underline{P}) = n\,log(2\beta\alpha) + \sum_{l=1}^{n} log\, g_{\underline{\Phi}}(z_l) + (\beta - 1)\sum_{l=1}^{n} log\, G_{\underline{\Phi}}(z_l)$$
$$-(\beta + 1)\sum_{l=1}^{n} log\, \overline{G}_{\underline{\Phi}}(z_l) - 2\sum_{l=1}^{n}\left[O_{\underline{\Phi}}(z_l)\right]^{\beta} \quad (5.1)$$
$$+(\alpha - 1)\sum_{l=1}^{n} log\left(1 - exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}\right),$$

where $O_{\underline{\Phi}}(z_l)=\frac{G_{\underline{\Phi}}(z_l)}{\overline{G}_{\underline{\Phi}}(z_l)}$. The TLL can indeed be optimized urgently utilizing SAS software or the R-language, or implicitly by solving non-linear LL equations obtained by differentiating (5.1). The score function's related components $U_n(\psi)=\left(\frac{\partial L_n(\underline{P})}{\partial \alpha},\frac{\partial L_n(\underline{P})}{\partial \beta},\frac{\partial L_n(\underline{P})}{\partial \underline{\Phi}}\right)^T$ are

$$\frac{\partial L_n(\underline{P})}{\partial \alpha}=\frac{n}{\alpha}+\sum_{l=1}^{n} log\left(1-exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}\right),$$

$$\frac{\partial L_n(\underline{P})}{\partial \beta}=\frac{n}{\beta}\sum_{l=1}^{n} log\, G_{\underline{\Phi}}(z_l)-\sum_{l=1}^{n} log\, \overline{G}_{\underline{\Phi}}(z_l)-2\sum_{l=1}^{n}\left[O_{\underline{\Phi}}(z_l)\right]^{\beta} log\left[O_{\underline{\Phi}}(z_l)\right]$$
$$+2(\alpha-1)\sum_{l=1}^{n}\frac{log\left[O_{\underline{\Phi}}(z_l)\right]\left[O_{\underline{\Phi}}(z_l)\right]^{\beta} exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}}{1-exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}},$$

and

$$\frac{\partial L_n(\underline{P})}{\partial \underline{\Phi}_k}=\sum_{l=1}^{n}\frac{\partial G_{\underline{\Phi}}(z_l)/\partial \underline{\Phi}_k}{G_{\underline{\Phi}}(z_l)}+(\beta-1)\sum_{l=1}^{n}\frac{\partial G_{\underline{\Phi}}(z_l)/\partial \underline{\Phi}_k}{G_{\underline{\Phi}}(z_l)}+(\beta+1)\sum_{l=1}^{n}\frac{\partial G_{\underline{\Phi}}(z_l)/\partial \underline{\Phi}_k}{\overline{G}_{\underline{\Phi}}(z_l)}$$
$$-2\sum_{l=1}^{n} O_{\underline{\Phi}}(z_l)\,\partial G_{\underline{\Phi}}(z_l)/\partial \underline{\Phi}_k+2(\alpha-1)\sum_{l=1}^{n}\frac{exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}O_{\underline{\Phi}}(z_l)\partial G_{\underline{\Phi}}(z_l)/\partial \underline{\Phi}_k}{1-exp\left\{-2\left[O_{\underline{\Phi}}(z_l)\right]^{\beta}\right\}},$$

where, $\underline{\delta}_k$ is the k[th] member of the parameter vector δ. The ML estimation (MLE) of $\underline{P}$, is accomplished by solving nonlinear equations $U_n(\underline{P})=0$.

6. Applications to Real Data

This section examines two real-world data sets to demonstrate how capable of adapting the TLWE model is. The first data set [see Bjerkedal (1960)] is established as the "failure times data," and it consists of lifetime data on adsorption capacity of the adsorbent individuals' "relief times" (in minutes). Gross and Clark (1975) looked into and published the "survival times" in days for 72 guinea pigs infected with virulent tubercle bacilli in the second data set.

We will make a comparison the TLWExp distribution's fits to the following competing models: Exp, odd Li Exp (OLiExp), Marshall-Olkin (M-O) Exp (M-OExp), Moment Exp (MoExp), the logarithmic Burr-Hatke Exp (LBHExp), generalized M-OExp (M-OExp), beta Exp (BExp), M-O Kumaraswamy Exp (M-OKExp) (KM-OExp). See Refaie (2018) and Ibrahim (2018) for PDFs of the offerings that are compatible (2020).

We discuss the Anderson-Darling (m_1) and the Cramér-Von Mises (m_2) statistics, as well as the Tables 2 and 4 gives the MLEs, SErs, confidence intervals (Co.I.s) values for the both data sets. Tables 3 and 5 illustrate the m_1; $m_2$for the first and second data sets. Figures 3 and 4 gives the P-P plot (PPP), Kaplan-Meier plot (KMP), estimated PDF (EPDF) and ECDF for the both data sets.

Table 2. Results of MLEs, SErs, Co.I.s (in parentheses) for the competitive models for the first data

Models	Estimates, SEs and C.I.s
Exp(θ)	0.5261
	(0.1172)
	(0.3; 0.8)
MoExp(θ)	0.950
	(0.150)
	(0.7, 1.2)
LBHExp(θ)	0.5263
	(0.118)
	(0.4; 0.6)
OLiExp(θ)	0.6044
	(0.0535)
	(0.5; 0.7)
BXExp(a,θ)	1.1635; 0.321
	(0.33); (0.03)
	(0.5, 1.8); (0.26,0.4)
M-OExp(α,θ)	54.47, 2.32
	(35.58); (0.37)
	(0, 124.2); (1.58, 3.0)
TLWExp(α,β,θ)	**8.03, 1.58, 3.15**
	(4.22); (1.01); (0.025)
	(0, 16.5); (0, 3.6); (3.1, 3.2)
BExp(α,β,θ)	81.633; 0.542, 3.514
	(120.41); (0.327); (1.410)
	(0, 317.63); (0, 1.18); (0.75, 6.3)
KExp(α,β,θ)	83.756; 0.568, 3.330
	(42.361); (0.326); (1.188)
	(0.7, 167); (0, 1.2); (1.00, 5.7)
GM-OExp(λ,α,θ)	0.519, 89.462, 3.169
	(0.256); (66.278); (0.77)
	(0.02, 1.02); (0, 219.4); (1.66, 4.7)
KM-OExp(α,β,λ,θ)	8.868, 34.826; 0.299, 4.899
	(9.15); (22.31); (0.24); (3.18)
	(10.9, 46.8); (0, 78.6); (0; 0.76); (0, 11)
M-OKExp(α,β,λ,θ)	0.133, 33.232; 0.571, 1.669
	(0.332); (57.84); (0.72); (1.81)
	(0; 0.8); (0, 146.6); (0, 2); (0, 5.2)

Table 3. Results of m_1 and $m_2$for the first data.

Models	m_1	m_2
Exp	4.60	0.96
KExp	0.45	0.07
BXExp	1.33	0.24
M-OExp	0.80	0.14

GM-OExp	0.51	0.08
KM-OExp	1.08	0.19
M-OKExp	0.60	0.11
OLiExp	1.30	0.22
BExp	0.70	0.12
MoExp	2.76	0.53
LBHExp	0.62	0.11
TLWExp	0.36	0.040

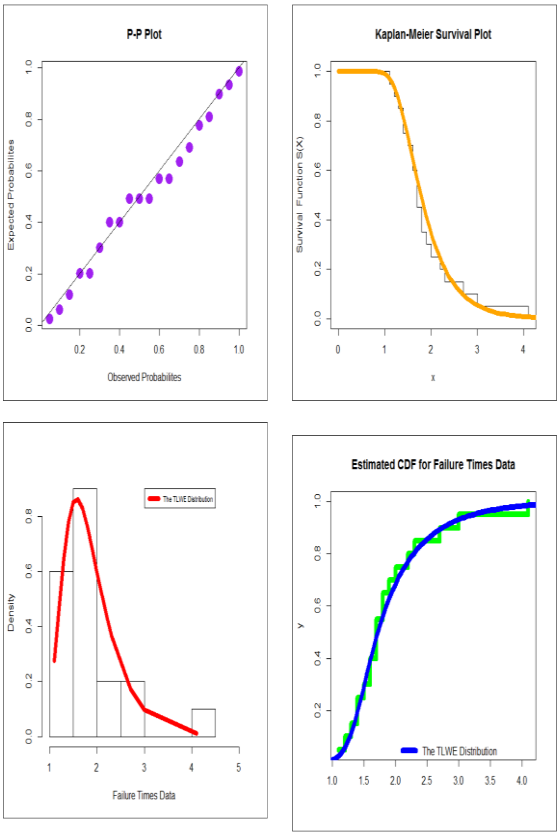

Figure 3. PPP, KMP, EPDF and ECDF for the first data.

Table 4. Results of MLEs, SErs, Co.I.s (in parentheses) for the competitive models for the second data

Models	MLEs, SErs and Co.I.s
Exp(b)	0.540
	(0.063)
	(0.4; 0.7)
OLiExp(θ)	0.38145
	(0.021)
	(0.3; 0.4)
MoExp(θ)	0.9250
	(0.080)
	(0.62, 1.08)
LBHExp(θ)	0.542
	(0.06)
	(0.41; 0.68)
BXExp(a, θ)	0.480; 0.2060
	(0.061); (0.012)
	(0.4; 0.5); (0.18; 0.23)

M-OExp(α,θ)	8.780, 1.380
	(3.555); (0.193)
	(1.81,15.74); (1.0,1.80)
TLWExp(α,β,θ)	3.225, 1.55; 0.018
	(0.85); (0.25); (0.059)
	(1.5, 4.9); (1, 2); (0; 0.136)
GM-OExp(λ,α,θ)	0.179, 47.635, 4.470
	(0.07); (44.901); (1.327)
	(0.04; 0.3); (0, 14); (2, 7)
KExp(a,β,θ)	3.3039, 1.101, 1.038
	(1.120); (0.763); (0.615)
	(1.12, 5.53); (0, 2.62); (0, 2.24)
M-OKExp(α,β,λ,θ)	0.008, 2.716, 1.986; 0.099
	(0.002), 1.316); (0.784); (0.048)
	(0.004,0.010); (0.14, 5); (0.4, 4); (0; 0.2)

Table 5. Results of m_1 and m_2 for the second data.

Models	m_1	m_2
Exp	6.53	1.25
M-OKExp	0.79	0.12
OLiExp	1.94	0.33
MoExp	1.52	0.25
LBHExp	0.79	0.19
GM-OExp	1.02	0.16
KExp	0.74	0.11
BXExp	2.90	0.52
M-OExp	1.20	0.17
TLWExp	0.75	0.12

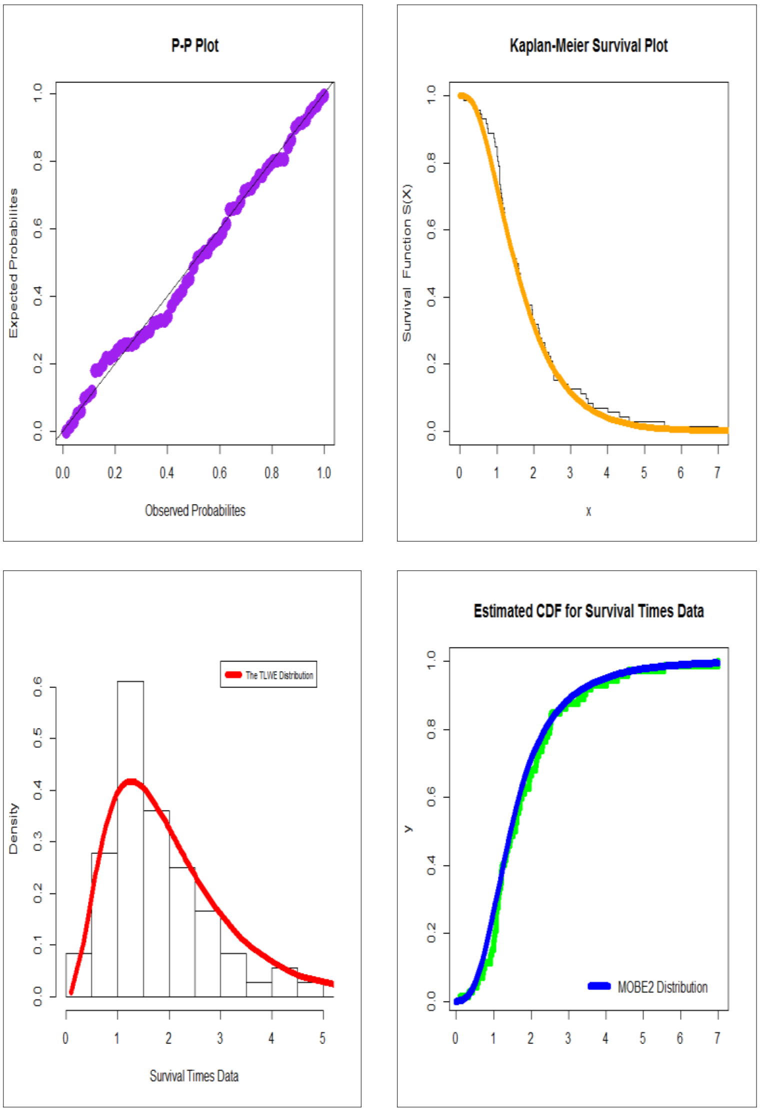

Figure 4. PPP, KMP, EPDF and ECDF for the second data.

Many widely known competitor models, such as the Exp, M-OExp, OLiExp, LBHExp, MoExp, GM-OExp, KExp, M-OKExp, and KM-OExp, outperform the TLWExp model. As a result, for both data sets, the new lifespan model offers an acceptable alternate solution to all of these models. Figures 3 and 4 demonstrate how well the TLWExp model fits the two real data sets.

7. Summary and concluding remarks

In this paper, we characterized and investigated the TLWG family, a novel generator of continuous lifespan distributions. The family's statistical characteristics, such as density function expansion, quantile function and moments are provided. Two applications to real-world data sets clearly show the desired family's significance and versatility.

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