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Research article

Statistical Properties and Applications of the Discrete Exponentiated Modified Topp-Leone Chen Distribution

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Abstract: In this paper, a new distribution with four parameters named discrete exponentiated modified Topp-Leone Chen distribution is introduced, using the general approach of discretization. The discrete exponentiated modified Topp-Leone Chen distribution can be applied to model reliability and survival data in various fields. Some statistical properties including quantile, mean residual life, mean time to failure, mean time between failure, availability, Rényi entropy, moments and order statistics are obtained. Maximum likelihood approach is applied under Type-II censored sample for estimating the unknown parameters of the discrete exponentiated modified Topp-Leone Chen distribution. Also, maximum likelihood estimators of the survival, hazard rate and alternative hazard rate functions are derived. The confidence intervals for the parameters, survival, hazard rate and alternative hazard rate functions are obtained. A simulation study is carried out to illustrate the theoretical results of the maximum likelihood estimation. Finally, two real data sets are applied to illustrate the flexibility and applicability of the proposed model for real-life applications.

Keywords: Survival discretization; Modified Topp-Leone Chen; Confidence intervals; Maximum likelihood method; Markov Chain Monte Carlo simulation method.

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1. Introduction

Discretization of continuous probability distributions has garnered considerable attention in recent years. Traditional discrete distributions like the geometric and Poisson, while widely used, may have limited applicability when modeling reliability, failure times or count data. This has led to the development of some discrete distributions based on popular continuous models for reliability, failure times, etc. Therefore, several discrete distributions have been presented in the literature. Chakraborty (2015) surveyed different methods for generating discrete analogues of continuous probability distributions, among these is the survival discretization method. If the underlying continuous failure time X has the *survival function* (sf), $S(x) = P(X \geq x)$ and times are grouped into unit intervals so that the discrete observed variable is *discrete* X $(dX) = \lfloor X \rfloor$, is the largest integer less than or equal to X, the *probability mass function* (pmf) of dX can be written as

$$\begin{aligned} P(x) &= P(dX = x) = P(x \leq dX < x + 1) \\ &= S(x) - S(x + 1), \qquad\qquad x = 0,1,2, \dots \end{aligned} \tag{1}$$

Therefore, for any continuous distribution, it is possible to construct corresponding discrete distribution using (1).
The discretization of a continuous distribution using this method retains the same functional form of the sf, $S(x)$. As a result, many reliability characteristics remain unchanged. This method, which is widely applied to generate new discrete distributions has received attention in the last four decades.

Many authors applied the general approach of discretization of some known continuous distributions for use as lifetime distributions. For example, Nakagawa and Osaki (1975) proposed a discrete Weibull distribution. Roy (2004) proposed a discrete Rayleigh distribution as a particular case of the discrete Weibull. Krishna and Pundir (2009) derived the discrete Burr XII distribution and applied it to fit the reliability in series system. Also, they derived the discrete Pareto distribution as a special case of the discrete Burr XII distribution. Gomez-Deniz and Calderin-Ojeda (2011) constructed the discrete version of Lindley distribution and used it as an alternative to Poisson distribution to model automobile claim frequency data. Nekoukhou *et al*. (2012) presented a version of the discrete generalized exponential distribution, which can be viewed as different generalization of the geometric distribution. Moreover, they discussed some of its distributional and moment properties. AL-Huniti and AL-Dayian (2012) proposed the discrete Burr Type III distribution. Also, they discussed some important properties and estimated the parameters based on the *maximum likelihood* (ML) and Bayesian approaches. Hegazy *et al*. (2018) presented the discrete Gompertz distribution. In addition, they discussed some statistical properties of the distribution and estimated the unknown parameters based on the ML method. Elmorshedy and Eliwa (2019) proposed a two-parameter exponentiated discrete Lindley distribution and studied some of its statistical properties of the distribution. They used the ML method to estimate the unknown parameters of the distribution.

Altun *et al*. (2022) introduced a study on discrete Bilal distribution with properties and applications on integer valued autoregressive process and they discussed the structural properties of the proposed distribution. Also, they used the ML and moments methods to estimate the unknown model parameters. Almetwally and Ibrahim (2020) proposed the discrete alpha power inverse Lomax distribution with application of COVID-19 data and discussed some of its statistical properties. They derived the ML estimators and confidence intervals for the parameters. Eliwa *et al*. (2020) introduced the discrete Gompertz-G Family of distributions for over-and under-dispersed data. Also, they studied some of its distributional properties and reliability characteristics. They used the ML method for estimating the family parameters. Chotedelok and Bodhisuwan (2020) obtained the discrete exponentiated Pareto distribution with properties and application.

Tyagi *et al*. (2020) presented inferences on discrete Rayleigh distribution under Type II censored data. Ibrahim *et al.* (2021) derived the discrete analogue of the Weibull-G family. They studied its properties and estimation the distribution parameters using Bayesian and non- Bayesian estimation methods. El-Deep *et al*. (2021) introduced discrete analog of inverted Topp-Leone distribution.
Eliwa *et al*. (2022) introduced a discrete exponential generalized-G family of distributions. A flexible probability tool for modeling extreme and zero inflated count data under different shapes of hazard rates. They derived and analyzed many relevant mathematical and statistical properties, among other they applied some classical estimation methods, including Cramér–von Mises, ordinary least squares, L-moments, ML, Kolmogorov, bootstrapping and weighted least squares. Additionally, they obtained the Bayes estimators for the parameters under the squared error loss function. They explained the usefulness of the class by using four real data sets.

Eliwa *et al*. (2023) presented *univariate* probability-G classes for scattered samples under different forms of hazard: continuous and discrete version with their inference's tests. They defined a generator to propose continuous as well as discrete families of distributions. They discussed some mathematical and statistical properties of these G-classes and described some structural properties of two special models of these classes.

Abd EL-Hady *et al*. (2023) derived *Discrete exponentiated-Generalized* family of distributions with parameter α, denoted by DE-G family(α)distributions. They obtained some statistical properties and the estimation of its unknown parameters using the ML method. The corresponding pmf of DE-G family is given by

$$P(x;\alpha) = [G(x+1)]^{\alpha} - [G(x)]^{\alpha}, \qquad x = 0,1,2,\dots, \qquad (\alpha > 0). \tag{2}$$

The *cumulative distribution function* (cdf), sf, *hazard rate function* (hrf) and *alternative* hrf (ahrf) can be formulated as:

$$F(x;\alpha) = [G(x+1)]^{\alpha}. \qquad x = 0,1,2,\dots, \qquad (\alpha > 0), \tag{3}$$

$$S(x;\alpha) = 1 - [G(x)]^{\alpha}, \qquad x = 0,1,2,\dots, \qquad (\alpha > 0), \tag{4}$$

$$h(x;\alpha) = \frac{[G(x+1)]^{\alpha} - [G(x)]^{\alpha}}{1 - [G(x)]^{\alpha}}, \qquad x = 0,1,2,\dots, \qquad (\alpha > 0), \tag{5}$$

and

$$ah(x;\alpha) = ln\left[\frac{1-[G(x)]^{\alpha}}{1-[G(x+1)]^{\alpha}}\right], \qquad x = 0,1,2,..., \qquad (\alpha > 0). \qquad (6)$$

Adding one or more parameters to a distribution makes the resulting distribution richer and more flexible for analyzing and modeling data.

Therefore, AL-Sayed *et al.* (2022) introduced the *modified Topp-Leone Chen* (MTLCh) distribution as a composite distribution using the transformation $X = \frac{T}{\beta}$, where T is a rv. They obtained some statistical properties of the proposed distribution. They derived the ML estimators of the unknown parameters under progressive Type-II censored samples. They gave a numerical example to illustrate the theoretical results and used two real data sets to demonstrate how the results can be used in practice. The *probability density function* (pdf) and cdf of the MTLCh distribution are, respectively, given by

$$f(x;\theta,\lambda,\beta) = 2\lambda\theta\beta e^{[\beta x+2\lambda(1-e^{\beta x})]}\left[1-exp\left(2\lambda\left(1-e^{\beta x}\right)\right)\right]^{\theta-1}, \; x>0, (\theta,\lambda,\beta>0), \qquad (7)$$

and

$$F(x;\theta,\lambda,\beta) = \left[1-exp\left(2\lambda\left(1-e^{\beta x}\right)\right)\right]^{\theta}, \qquad x>0, (\theta,\lambda,\beta>0), \qquad (8)$$

where λ, θ are shape parameters and β is a scale parameter.

This paper is organized as follows: *a discrete exponentiated- MTLCh* (DE-MTLCh) distribution is introduced and some of its statistical properties are derived in Section 2. While, in Section 3, ML estimators are derived for the unknown parameters. Simulation study and results are presented in Section 4. Section 5 introduced two real data sets to show the applicability and flexibility of the DE-MTLCh. Conclusion is discussed in Section 6.

2. Construction of Discrete Exponentiated Modified Topp-Leone Chen Distribution

By substituting the cdf of the MTLCh distribution given by (8) into the general expression for the pmf (2), the pmf of the DE-MTLCh distribution can be obtained as:

$$p(x;\theta,\beta,\alpha,\lambda) = \left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}, \; x = 0,1,2,...,$$
$$(\theta,\alpha,\lambda,\beta > 0). \qquad (9)$$

The cdf, sf, hrf and ahrf of the DE-MTLCh distribution are given by

$$F(x;\theta,\beta,\alpha,\lambda) = \left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}, \quad x = 0,1,2,..., \quad (\theta,\alpha,\lambda,\beta>0), \qquad (10)$$

$$S(x;\theta,\beta,\alpha,\lambda) = 1-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}, \quad x = 0,1,2,..., \quad (\theta,\alpha,\lambda,\beta>0), \qquad (11)$$

$$h(x;\theta,\beta,\alpha,\lambda)=\frac{\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}}{1-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}},\ x=0,1,2,\dots,(\theta,\alpha,\lambda,\beta>0), \quad (12)$$

and

$$ah(x;\theta,\beta,\alpha,\lambda)=\ln\left[\frac{1-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}}{1-\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}}\right],\qquad x=0,1,2,\dots,(\theta,\alpha,\lambda,\beta>0). \quad (13)$$

There is a relationship between $ah(x)$ and $h(x)$, given by:
$h(x)=1-e^{-ah(x)}$.

The two concepts $h(x)$ and $ah(x)$ have the same monotonic property, i.e., $ah(x)$ is increasing (decreasing) if and only if $h(x)$ is increasing (decreasing).

Plots of pmf, hrf and ahrf of DE-MTLCh distribution are presented in Figures 1-3, for some selected values of the parameters. Figure 1 shows that the pmf of the

DE-MTLCh distribution can be unimodal, decreasing and increasing according to the selected value of the parameters. Figures 2 and 3 indicate that the hrf and ahrf of the DE-MTLCh distribution are decreasing, bathtub and increasing depending on the value of the parameters.

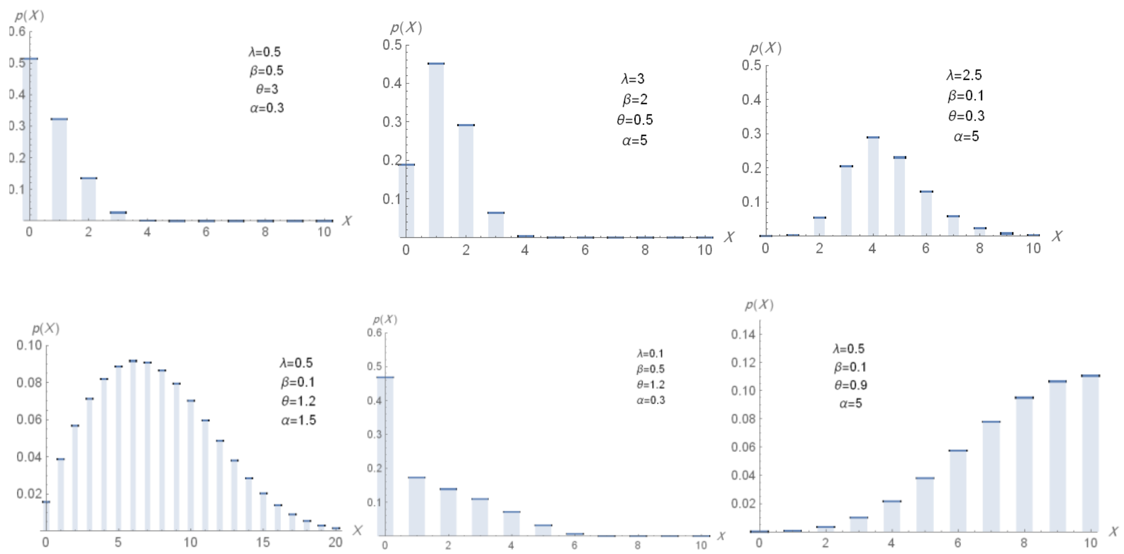

Figure 1. Plots of the pmf of the DE-MTLCh for different values of $\lambda,\beta,\theta,\alpha$

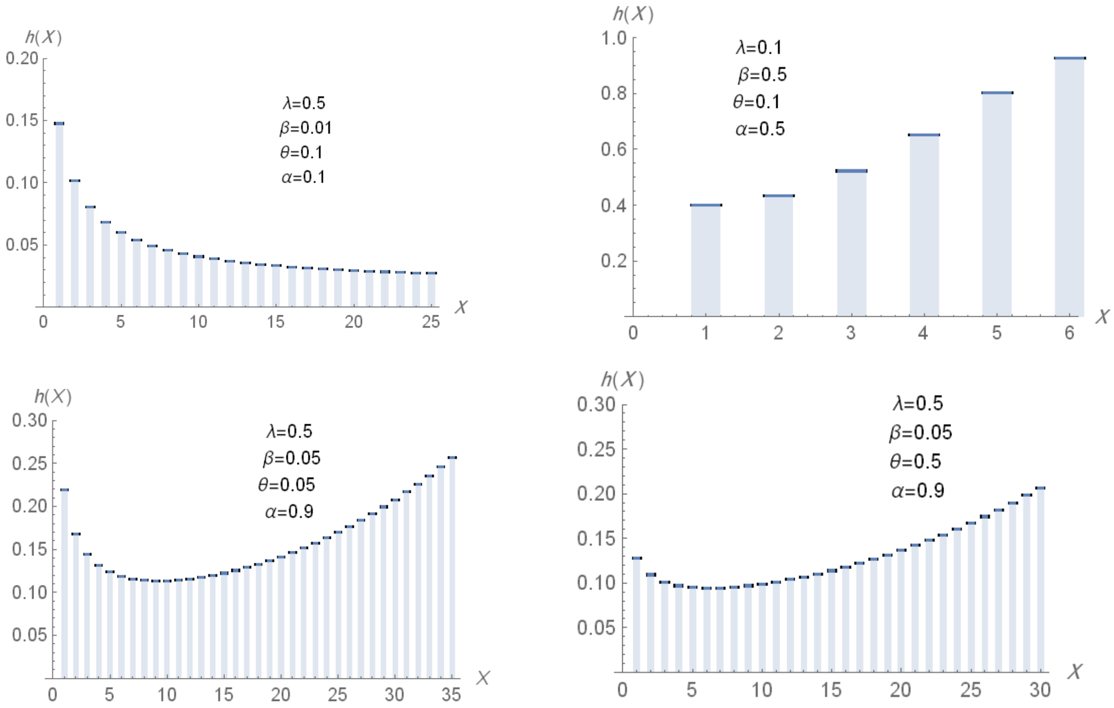

Figure 2. Plots of the hrf the DE-MTLCh for different values of $\theta, \alpha, \lambda, \beta$

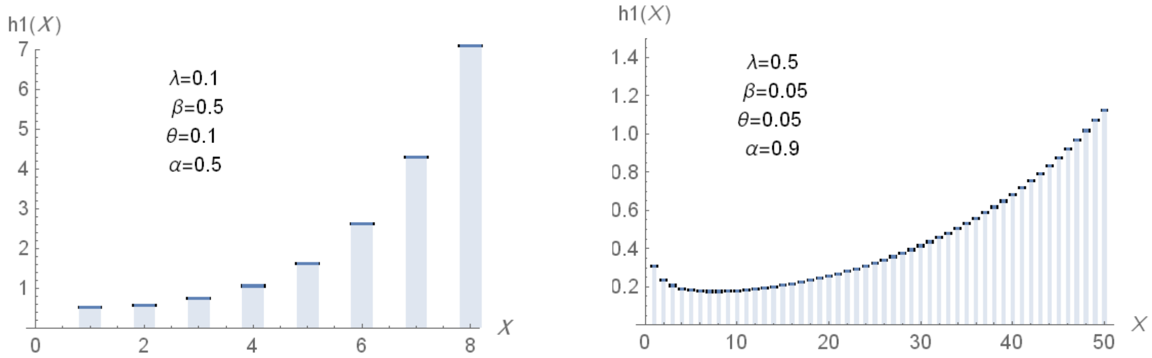

Figure 3. Plots of the ahrf of the DE-MTLCh for different values of $\theta, \alpha, \lambda, \beta$

2.1 Some statistical properties of discrete exponentiated-modified Topp-Leone Chen distribution

This subsection is devoted to obtain some important statistical properties of the DE-MTLCh distribution, such as the quantile function, moments, order statistic, Rényi entropy, mean residual lifetime(MRL), Mean time to failure (MTTF), Mean time between failure (MTBF) and availability(Av).

2.1.1 Quantile function

The u^{th} quantile x_u, of the DE-MTLCh distribution is given by

$$x_u = \left\lceil \frac{1}{\beta} \log\left[1 - \frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right] - 1 \right\rceil, \qquad 0 < u < 1, \tag{14}$$

where $\lceil X \rceil$ denotes the smallest integer greater than or equal to X.

Proof

$p(X \leq x_u) \geq u,$ from (10)

$$\left[1-e^{2\lambda\left(1-e^{\beta(x_u+1)}\right)}\right]^{\alpha\theta} \geq u,$$

$$\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right] \geq u^{1/\alpha\theta}, \qquad \left(1-u^{1/\alpha\theta}\right) \geq \left(e^{2\lambda\left(1-e^{\beta(x_u+1)}\right)}\right),$$

$$\log\left(1-u^{1/\alpha\theta}\right) \geq 2\lambda\left(1-e^{\beta(x_u+1)}\right), \quad \frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda} \geq \left(1-e^{\beta(x_u+1)}\right),$$

$$e^{\beta(x_u+1)} \geq 1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}.$$

Hence

$$x_u = \left[\frac{1}{\beta}\log\left[1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right]-1\right], \qquad 0<u<1. \tag{15}$$

Similarly, if $p(X \geq x_u) \geq 1-u,$ one obtains

$$x_u \leq \frac{1}{\beta}\log\left[1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right], \qquad 0<u<1. \tag{16}$$

Combining (15) and (16), one gets

$$\frac{1}{\beta}\log\left[1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right]-1 \leq x_u \leq \frac{1}{\beta}\log\left[1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right].$$

Hence, x_u is an integer value given by

$$x_u = \left[\frac{1}{\beta}\log\left[1-\frac{\log\left(1-u^{1/\alpha\theta}\right)}{2\lambda}\right]-1\right], \qquad 0<u<1, \tag{17}$$

by putting $u = 0.5$ in (17), one gets the median of the DE-MTLCh distribution as follows

$$x_{0.5} = \left[\frac{1}{\beta}\log\left[1-\frac{\log\left(1-(0.5)^{1/\alpha\theta}\right)}{2\lambda}\right]-1\right], \qquad 0<u<1.$$

2.1.2 The moments of the discrete exponentiated modified Topp-Leone Chen distribution

a. The non-central moments

The non-central moments of the DE-MTLCh distribution are

$$\mu_r' = \sum_x x^r\left\{\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right\}, \qquad r = 1,2,.... \tag{18}$$

In particular, the mean μ is given by:

$$\mu_1' = \mu = \sum_x x\left\{\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right\}, \tag{19}$$

The second non-central moments is given by:

$\mu'_2 = \sum_{x \in R} x^2 [p(x_i)].$

b. The central moments

The central moments can be obtained by using the relation between the central and non-central moments as given below.

$\mu = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu'_{r-j}, \quad r = 1,2, \ldots,$

The variance of DE-MTLCh distribution can be obtained as follows:

$$\mu_2 = \mu'_2 - \mu^2$$

$$\mu_2 = \sum_x x^2 \left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right] - \left[\sum_x x \left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right]\right]^2.$$

(20)

c. The standard moments

The r^{th} standard moments can be obtained as follows:

$$\tilde{\mu}_r = E\left(\frac{X-\mu}{\sigma}\right)^r.$$

(21)

The skewness (Sk) and kurtosis (Kur) are, respectively, given by

$$\tilde{\mu}_3 = \frac{\mu_3}{\mu_2^{3/2}} \quad \text{and} \quad \tilde{\mu}_4 = \frac{\mu_4}{\mu_2^2}, \quad \text{where} \quad \tilde{\mu}_r = E(X-\mu)^r,\ r = 1,2, \ldots. \tag{22}$$

The index of dispersion (ID) is defined as the variance-to-mean ratio or dispersion index, it is a statistical measure used to quantify whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model it indicates.

If **ID = 1:** The data follows a Poisson distribution, indicating random occurrence.

If **ID < 1:** The data is under-dispersed, indicating a more regular pattern than random.

If **ID > 1:** The data is over-dispersed, indicating clustering or clumping.

The ID and coefficient of variation (CV) for the DE-MTLCh distribution can be obtained as

$$ID = \frac{\text{Variance}}{\text{Mean}} = \frac{\mu_2}{\mu}, \qquad \text{and} CV = \frac{(\mu_2)^{1/2}}{\mu}. \tag{23}$$

The mean, variance, Sk, Kur, ID and CV of the DE-MTLCh distribution for different values of the parameters are calculated numerically and displayed in Table 1using (19)-(23).

Table 1
Some descriptive measures of the DE-MTLCh distribution
for different values of the parameters

parameters				Descriptive Measures					
α	θ	β	λ	Mean	Variance	Kur	Sk	ID	CV
0.3	**0.9**	**0.5**	**0.5**	0.2418	0.3236	9.6727	2.5685	1.3382	2.3525
0.7				0.5101	0.5684	4.3499	1.3985	1.1142	1.4779
0.9				0.6250	0.6384	3.5818	1.1188	1.0214	1.2783
0.3	**0.9**	**0.5**	**0.1**	1.0081	2.0507	3.9865	1.3545	2.0342	1.4205
			0.6	0.1932	0.2454	11.0845	2.7929	1.2701	2.5640
			0.8	0.1300	0.1527	14.1052	3.2390	1.1730	3.0059
0.3	**0.5**	**1.5**	**0.8**	1.3279	1.1404	2.5503	0.4684	0.8587	0.8031
		2		1.5697	1.1027	2.5475	0.2883	0.7024	0.6692
		2.5		1.7597	1.0410	2.6201	0.1899	0.5915	0.5798

From Table 1, it can be observed that as the values of parameters α increase, the mean and variance of the DE-MTLCh distribution also increase. As λ increases, the mean and variance decreases, furthermore as θ decreases the mean and variance increase. The DE-MTLCh distribution exhibit positive Sk which implies that the tail of the distribution is longer on the right side. For the Kur value (Kur > 3) indicates leptokurtosis distribution, while a value (Kur < 3) shows a platykurtic distribution. Additionally, the mean of the distribution can be either smaller or larger than the variance, making it suitable for modeling both over-dispersed and under-dispersed data. The varying CV values suggest different levels of variability across the datasets. Lower CV values indicate less dispersion around the mean, while higher CV values suggest greater variability.

2.1.3 The order statistics of the discrete exponentiated-modified Topp-Leone Chen distribution

Order statistics play an important role in various fields of statistical theory and practice. The cdf of the i^{th} order statistic of the DE-MTLCh is given by

$$F_{i:n}(x;\theta,\beta,\alpha,\lambda) = \sum_{r=i}^{n} \binom{n}{r} F[(x;\theta,\beta,\alpha,\lambda)]^r [1 - F(x;\theta,\beta,\alpha,\lambda)]^{n-r}. \quad (24)$$

Using the binomial expansion for $[1 - F(x;\theta,\beta,\alpha,\lambda)]^{n-r}$ and substituting (10) in (24) it follows that

$$F_{DE_{i:n}}(x;\theta,\beta,\alpha,\lambda)$$

$$= \sum_{r=i}^{n} \binom{n}{r} [F(x;\theta,\beta,\alpha,\lambda)]^r \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x;\theta,\beta,\alpha,\lambda)]^j,$$

$$F_{DE_{i:n}}(x;\theta,\beta,\alpha,\lambda) = \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]\right]^{\alpha\theta(r+j)}. \quad (25)$$

Special cases

Case I: if $i = 1$ in (25), one can obtain the distribution function of the first order statistics, as given below

$$F_{DE_1}(x;\theta,\alpha,\lambda,\beta) = 1 - [1 - F_{DE}(x;\theta,\alpha,\lambda,\beta)]^n$$

$$= 1 - \left[1 - \left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta}\right]^n. \quad (26)$$

Case II: if $i = n$ in (25), the cdf of the largest order statistics, it as follows:

$$F_{DE_n}(x;\theta,\alpha,\lambda,\beta) = [F(x;\theta,\alpha,\lambda,\beta)]^n = \left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{n\alpha\theta}. \quad (27)$$

The corresponding pmf of the i^{th} order statistics can be expressed as

$$P\left(X_{(i)} = x\right) = \frac{n!}{(i-1)!(n-i)!}\int_{F(x-1)}^{F(x)} v^{i-1}(1-v)^{n-i}dv, \quad (28)$$

where v is a rv.

Using the binomial expansion for $(1-v)^{n-i}$ then, the pmf of (28) is

$$P_{DE}\left(X_{(i)} = x\right) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\int_{F(x-1)}^{F(x)} v^{i+j-1}dv$$

$$= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left(\frac{1}{i+j}\right)$$

$$\times\left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta(i+j)} - \left[1 - e^{2\lambda\left(1-e^{\beta(x)}\right)}\right]^{\alpha\theta(i+j)}\right]. \quad (29)$$

See, Arnold et al. (2008).

The pmf of the smallest order statistics is obtained by substituting $i = 1$ in (28) as given below

$$P_{DE}\left(X_{(1)} = x\right) = \frac{n!}{(1-1)!(n-1)!}\int_{F(x-1)}^{F(x)} v^{1-1}(1-v)^{n-1}\,dv$$

$$= 1 - \left[\left[1 - e^{2\lambda\left(1-e^{\beta(x)}\right)}\right]^{n\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{n\alpha\theta}\right]. \quad (30)$$

Also, the pmf of the largest order statistic is obtained by substituting $i = n$ in (28) as follows:

$$P_{DE}\left(X_{(n)} = x\right) = \left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{n\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta(x)}\right)}\right]^{n\alpha\theta}\right]. \quad (31)$$

2.1.4 Rényi entropy

Entropy refers to the amount of uncertainty associated with the rv. It has many applications in several fields such as econometrics, information theory, survival analysis and computer science [see, Rényi (1961)].

The measure of variation of the uncertainty of the discrete rv (drv) X can be expressed as

$$I_\eta(x) = \frac{1}{1-\eta}\log\sum_x\left[\left[1 - e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right]^\eta, \quad x = 0,1,2,\ldots. \quad (32)$$

The Shannon entropy can be defined by

$$I(X) = -\sum_{x}\left[\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right]$$

$$\times \log\left[\left[1-e^{2\lambda\left(1-e^{\beta(x+1)}\right)}\right]^{\alpha\theta} - \left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right], \qquad x = 0,1,2,\ldots \qquad (33)$$

The Shannon entropy can be derived as a particular case of the Rényi entropy when $\eta \to 1$.

2.1.5.1 Mean residual lifetime function, mean time to failure, mean time between failure and availability

Kemp (2004) defined the MRL as

$$MRL(x_0) = \frac{\sum_{k=x_0+1}^{\infty} s(k)}{s(x_0)}$$

$$= \frac{1}{1-\left[1-e^{2\lambda\left(1-e^{\beta(x_0)}\right)}\right]^{\alpha\theta}} \sum_{k=x_0+1}^{\infty}\left(1-\left[1-e^{2\lambda\left(1-e^{\beta(k)}\right)}\right]^{\alpha\theta}\right). \qquad (34)$$

MTTF, MTBF and Av are reliability terms based on methods and procedures for lifecycle predictions for a product. MTTF, MTBF and Av are ways of providing a numeric value based on a compilation of data to quantify a failure rate and the resulting time of expected performance. In addition, in request to design and manufacture a maintainable system, it is necessary to predict the MTTF, MTBF and Av. [see, Eliwa et al. (2020)].

The MTTF is given as

$$MTTF = \sum_{x=1}^{\infty} S(x) = \sum_{x=1}^{\infty} 1-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}, \qquad x > 0. \qquad (35)$$

Then the MTBF is given as

$$MTBF = \frac{-x}{\log[S_{DE}(x)]} = \frac{-x}{\log\left[1-\left[1-e^{2\lambda\left(1-e^{\beta x}\right)}\right]^{\alpha\theta}\right]}, \qquad x > 0. \qquad (36)$$

Av is considered as being the probability that the component is successful at time t,

$$Av = \frac{MTTF}{MTBF}. \qquad (37)$$

3. Maximum Likelihood Estimation

This section is devoted to estimate the vector of the parameters, $\underline{\varphi} = (\theta, \alpha, \beta, \lambda)$, sf, hrf and ahrf of the DE-MTLCh $(\theta, \alpha, \beta, \lambda)$distribution, based on Type-II censored samples, also confidence intervals of the parameters $\theta, \alpha, \beta, \lambda$,sf, hrf and ahrf are derived.

Suppose that $x_1, x_2, \ldots, x_r$ is a Type-II censored sample of size r obtained from a life-test on n items whose lifetimes have a DE-MTLCh $(\theta, \alpha, \beta, \lambda)$distribution. Then the likelihood function is

$$L_{DE}\left(\underline{\varphi}; \underline{x}\right) \propto \{\prod_{i=1}^{r} P(x_i)\}[S_{DE}(x_r)]^{n-r}, \quad (38)$$

where $P_{DE}(x)$ and $S_{DE}(x)$ are given, respectively, by (9) and (11).

$$L_{DE}\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r}\left(\left[1 - e^{2\lambda\left(1-e^{\beta(x_i+1)}\right)}\right]^{\alpha\theta} - \left[1 - e^{2\lambda\left(1-e^{\beta(x_i)}\right)}\right]^{\alpha\theta}\right)\right\},$$

$$\times\left[1 - \left[1 - e^{2\lambda\left(1-e^{\beta(x_r)}\right)}\right]^{\alpha\theta}\right]^{n-r},$$

$$L_{DE}\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r}\left([w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}\right)\right\}\left[1 - [w_r]^{\alpha\theta}\right]^{n-r},$$

where

$$w_{i1} = \left[1 - e^{2\lambda\left(1-e^{\beta(x_i+1)}\right)}\right],\ w_{i2} = \left[1 - e^{2\lambda\left(1-e^{\beta(x_r)}\right)}\right] \text{ and } w_r = \left[\left(1 - e^{2\lambda\left(1-e^{\beta x_r}\right)}\right)\right]. \quad (39)$$

The natural logarithm of the likelihood function is given by

$$\ell_{DE} \equiv lnL_{DE}\left(\underline{\varphi}; \underline{x}\right) \propto ln\left\{\prod_{i=1}^{r}\left([w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}\right)\left[1 - [w_r]^{\alpha\theta}\right]^{n-r}\right\}$$

$$\propto \sum_{i=1}^{r} ln\{[w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}\} + (n-r)ln\left[1 - [w_r]^{\alpha\theta}\right]. \quad (40)$$

Considering the four parameters θ, α, β and λ are unknown and differentiating the ln likelihood function in (40), with respect to θ, α, β and λ one can obtain

$$\frac{\partial\ell_{DE}}{\partial\theta} = \sum_{i=1}^{r}\left\{\frac{[[w_{i1}]^{\alpha\theta}(\alpha)ln(w_{i1}) - [w_{i2}]^{\alpha\theta}(\alpha)ln(w_{i2})]}{[[w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}]}\right\} - (n-r)\frac{[w_r]^{\alpha\theta}(\alpha)ln(w_r)}{[1-[w_r]^{\alpha\theta}]}, \quad (41)$$

$$\frac{\partial\ell_{DE}}{\partial\alpha} = \sum_{i=1}^{r}\left\{\frac{[[w_{i1}]^{\alpha\theta}(\theta)ln(w_{i1}) - [w_{i2}]^{\alpha\theta}(\theta)ln(w_{i2})]}{[[w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}]}\right\} - (n-r)\frac{[w_r]^{\alpha\theta}(\theta)ln(w_r)}{[1-[w_r]^{\alpha\theta}]}, \quad (42)$$

$$\frac{\partial\ell_{DE}}{\partial\beta} = \sum_{i=1}^{r}\left\{\frac{[(\alpha\theta[w_{i1}]^{\alpha\theta-1})\acute{w}_{i1} - [(\alpha\theta[w_{i2}]^{\alpha\theta-1})\acute{w}_{i2}]}{[[w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}]}\right\} - (n-r)\frac{(\alpha\theta(w_r)^{\alpha\theta-1})\acute{w}_r}{[1-[w_r]^{\alpha\theta}]}, \quad (43)$$

where

$$\acute{w}_{i1} = \left[e^{2\lambda\left(1-e^{\beta(x_i+1)}\right)}\right]\left(2\lambda\, e^{\beta(x_i+1)}\right)(x_i+1),$$

$$\acute{w}_{i2} = \left[e^{2\lambda\left(1-e^{\beta(x_i)}\right)}\right]\left(2\lambda\, e^{\beta(x_i)}\right)(x_i),$$

and

$$\acute{w}_r = \left[e^{2\lambda\left(1-e^{\beta(x_r)}\right)}\right]\left(2\lambda\, e^{\beta(x_r)}\right)(x_r).$$

$$\frac{\partial\ell_{DE}}{\partial\lambda} = \sum_{i=1}^{r}\left\{\frac{[(\alpha\theta[w_{i1}]^{\alpha\theta-1})\acute{w}_{i1*} - (\alpha\theta[w_{i2}]^{\alpha\theta-1})\acute{w}_{i2*}]}{[[w_{i1}]^{\alpha\theta} - [w_{i2}]^{\alpha\theta}]}\right\} - (n-r)\frac{(\alpha\theta(w_r)^{\alpha\theta-1})\acute{w}_{r*}}{[1-[w_r]^{\alpha\theta}]}, \quad (44)$$

where

$$\acute{w}_{i1*} = \left[-e^{2\lambda\left(1-e^{\beta(x_i+1)}\right)}\right]2\left(1 - e^{\beta(x_i+1)}\right).$$

$\acute{w}_{i2*} = \left[-e^{2\lambda\left(1-e^{\beta(x_i)}\right)}\right] 2\left(1 - e^{\beta(x_i)}\right).$

and

$\acute{w}_{r*} = \left[-e^{2\lambda\left(1-e^{\beta(x_r)}\right)}\right] 2\left(1 - e^{\beta(x_r)}\right).$

Then the ML estimators of the parameters denoted by $\hat{\theta}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are derived by equating the nonlinear likelihood equations (41-44) to zeros and solving numerically.

Depending on the invariance property, the ML estimators of $S_{DE}(x)$, $h_{DE}(x)$and $ah_{DE}(x)$ can be obtained by replacing α,θ, β and λ with their corresponding ML estimators $\hat{\theta}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$, respectively in (11), (12) and (13), as given below

$$\hat{S}_{DE_{ML}}(x; \hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) = 1 - \left[1 - e^{2\hat{\lambda}\left(1-e^{\hat{\beta}x}\right)}\right]^{\hat{\alpha}\hat{\theta}}, \qquad x = 0,1,2, \ldots, \tag{45}$$

$$\hat{h}_{DE_{ML}}(x; \hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) = \frac{\left[1-e^{2\hat{\lambda}\left(1-e^{\hat{\beta}(x+1)}\right)}\right]^{\hat{\alpha}\hat{\theta}} - \left[1-e^{2\hat{\lambda}\left(1-e^{\hat{\beta}x}\right)}\right]^{\hat{\alpha}\hat{\theta}}}{1-\left[1-e^{2\hat{\lambda}\left(1-e^{\hat{\beta}x}\right)}\right]^{\hat{\alpha}\hat{\theta}}}, \quad x = 0,1,2, \ldots, \tag{46}$$

and

$$\widehat{ah}_{DE\,ML}(x; \hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}) = ln\left[\frac{1-\left[1-e^{2\hat{\lambda}\left(1-e^{\hat{\beta}x}\right)}\right]^{\hat{\alpha}\hat{\theta}}}{1-\left[1-e^{2\hat{\lambda}\left(1-e^{\hat{\beta}(x+1)}\right)}\right]^{\hat{\alpha}\hat{\theta}}}\right], \qquad x = 0,1,2, \ldots. \tag{47}$$

When the sample size is large and the regularity conditions are satisfied, the asymptotic distribution of the ML estimators is $\underline{\hat{\varphi}} \sim Bivariate\ Normal\left(\underline{\varphi}, I^{-1}\left(\underline{\varphi}\right)\right)$, where $\underline{\varphi} = (\theta, \alpha, \beta, \lambda)$, $\underline{\hat{\varphi}} = \left(\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right)$ and $I^{-1}\left(\underline{\varphi}\right)$ is the asymptotic variance covariance matrix of the ML estimators of the parameters α, θ, β and λ which is the inverse of the asymptotic observed Fisher information matrix. The asymptotic observed Fisher information matrix can be obtained as follows:

$$I\left(\underline{\varphi}\right) \approx -\left[\frac{\partial^2 \ell_{DE}}{\partial\varphi_i \partial\varphi_j}\right]\Bigg|_{\underline{\varphi}=\underline{\hat{\varphi}},} \qquad i,j = 1,2,,3,4. \tag{48}$$

The asymptotic $100\,(1-\tau)\%$ confidence intervals for $\theta, \alpha, \beta, \lambda, S_{DE}(x)$, $h_{DE}(x)$and $ah_{DE}(x)$are given, respectively, by

$$L_{\varphi} = \hat{\varphi} - Z_{\frac{\tau}{2}}\, \sigma_{\hat{\varphi}}\,. \qquad \text{and} \qquad U_{\varphi} = \hat{\varphi} + Z_{\frac{\tau}{2}}\, \sigma_{\hat{\varphi}}\,. \tag{49}$$

where L_{φ} and U_{φ} are the *lower limit(LL)* and *upper limit* (*UL)* respectively, $\hat{\varphi}$ is $\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{S}_{DE}(x)$, $\hat{h}_{DE}(x)$ or $\widehat{ah}_{DE}\,(x)$, where Z is the $100\,\%\left(1 - \frac{\tau}{2}\right)$ the standard normal percentile, $(1-\tau)$ is confidence coefficient and $\sigma_{\hat{\varphi}}$ is the standard deviation.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results based on simulated and real data**.**

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on generated data from the DE-MTLCh distribution. The ML averages of the estimates, sf, hrf and ahrf based on complete sample and Type-II censoring are computed. Moreover, confidence intervals of the parameters, sf, hrf and ahrf are calculated. The simulation study is performed using Mathematica11.

Tables 2 shows the averages of the ML estimates, *relative absolute biases* (RABs), *mean square errors* (MSEs), variances of the parameters and 95% confidence intervals under three levels of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 60%, 80% and 100%. Table 3 displays the same computational results, but for different population parameter values from the DE- MTLCh distribution for different samples of size n, where ($n = 30, 60 and\ 120$) and *number of replications* (NR)$= 1000$.

The averages, RABs, variances of the ML estimates of the parameters, sf, hrf and ahrf are computed as follows:

1) Averages $= \frac{\sum_{i=1}^{NR} \boldsymbol{estimates}}{\boldsymbol{NR}}$
2) RABs (estimate) = $\frac{|\boldsymbol{bias}(\boldsymbol{estimate})|}{\boldsymbol{true\ value}}$
3) Variances (estimate) = ER (estimate) $-$ $bias^2$(estimate)
4) Estimated risk (estimate) $= \frac{\sum_{i=1}^{NR}(\boldsymbol{estimated - true\ value})^2}{\boldsymbol{NR}}$

Table 2:
Averages, relative absolute biases, mean square errors, variances of the ML estimates, 95% confidence intervals of the parameters, survival, hazard rate and alternative hazard rate functions at ($x_0 = 1$) from the DE-MTLCh distribution for different samples of size n, censoring size r, $NR = 1000$, ($\theta = 0.5$, $\alpha = 0.6$, $\beta = 0.037, \lambda = 0.01$)

n	r	Parameters	Average	RAB	MSE	Variance	UL	LI	length
30	18	θ	0.5564	0.1127	0.0048	0.0016	0.6364	0.4763	0.1601
		α	0.6676	0.3352	0.0069	0.0024	0.7637	0.5715	0.1921
		β	0.0430	0.1637	0.0001	0.0001	0.0637	0.0223	0.0413
		λ	0.0121	0.2116	0.00001	0.00001	0.0188	0.0053	0.0135
		$S_{DE}(x_0)$	0.9195	0.0397	0.0017	0.0005	0.9653	0.8737	0.0915
		$h_{DE}(x_0)$	0.0258	0.1705	0.00005	0.000024	0.0356	0.0160	0.0195
		$ah_{DE}(x_0)$	0.0261	0.1723	0.00005	0.000026	0.0362	0.0161	0.0200
	24	θ	0.5506	0.1012	0.0040	0.0014	0.62503	0.4762	0.1487
		α	0.6607	0.3148	0.0057	0.0020	0.75004	0.5714	0.1785
		β	0.0392	0.0597	0.00002	0.00002	0.04838	0.0300	0.0183
		λ	0.0113	0.1396	9.05×10^{-6}	7.10×10^{-6}	0.0166	0.0061	0.0104
		$S_{DE}(x_0)$	0.9190	0.0392	0.0018	0.00061	0.9676	0.8705	0.0970
		$h_{DE}(x_0)$	0.0251	0.1901	0.00006	0.000026	0.0353	0.0150	0.0202
		$ah_{DE}(x_0)$	0.0255	0.1921	0.0000	0.000029	0.0359	0.0151	0.0207
	30	θ	0.5414	0.0828	0.0030	0.0013	0.6123	0.4704	0.1418
		α	0.6497	0.3001	0.0043	0.0018	0.7348	0.5645	0.1702
		β	0.0378	0.02413	7.97×10^{-6}	7.17×10^{-6}	0.0431	0.0326	0.0105
		λ	0.0107	0.0797	5.90×10^{-6}	5.27×10^{-6}	0.0152	0.0062	0.0089
		$S_{DE}(x_0)$	0.9147	0.0343	0.0015	0.00061	0.9632	0.8662	0.0970
		$h_{DE}(x_0)$	0.0257	0.1739	0.00005	0.000025	0.0355	0.0158	0.0196
		$ah_{DE}(x_0)$	0.0260	0.1758	0.00005	0.000028	0.0361	0.0159	0.02010
60	36	θ	0.5501	0.1002	0.0038	0.0013	0.6229	0.4773	0.1456
		α	0.6601	0.3418	0.0056	0.0019	0.7475	0.5727	0.1747
		β	0.0410	0.1098	0.00005	0.00004	0.0535	0.0285	0.0250
		λ	0.0117	0.1745	0.0000	0.00001	0.0183	0.0051	0.0132
		$S_{DE}(x_0)$	0.9175	0.0375	0.0016	0.0005	0.9615	0.8734	0.0880
		$h_{DE}(x_0)$	0.0258	0.1685	0.00004	0.00001	0.0344	0.0172	0.0172
		$ah_{DE}(x_0)$	0.0262	0.1704	0.00004	0.00002	0.0350	0.0173	0.0176
	48	θ	0.54114	0.0822	0.00284	0.0011	0.6077	0.4745	0.1332
		α	0.6493	0.3148	0.0041	0.0016	0.7293	0.5694	0.1598
		β	0.0383	0.0369	0.00001	9.97×10^{-6}	0.0445	0.0321	0.0123
		λ	0.01102	0.1029	8.27×10^{-6}	7.21×10^{-6}	0.0162	0.0057	0.0105
		$S_{DE}(x_0)$	0.91435	0.0339	0.0014	0.0005	0.9585	0.8701	0.0883
		$h_{DE}(x_0)$	0.0259	0.1659	0.00004	0.00001	0.0344	0.0174	0.0172
		$ah_{DE}(x_0)$	0.02630	0.1677	0.00004	0.00001	0.0350	0.0175	0.0174

Table 2 (Continued)

n	r	parameters	Average	RABs	MSE	Variance	UL	IL	Length
60	60	θ	0.5343	0.0686	0.00214	0.0009	0.5953	0.4732	0.1221
		α	0.6411	0.2921	0.0030	0.0013	0.7144	0.5678	0.1465
		β	0.0374	0.0109	3.82×10^{-6}	3.65×10^{-6}	0.0411	0.0336	0.0074
		λ	0.01061	0.0610	5.77×10^{-6}	5.40×10^{-6}	0.0151	0.0060	0.0091
		$S_{DE}(x_0)$	0.9109	0.0301	0.0011	0.00049	0.9543	0.8675	0.0860
		$h_{DE}(x_0)$	0.0263	0.1523	0.00004	0.00001	0.0348	0.0179	0.0168
		$ah_{DE}(x_0)$	0.0267	0.1541	0.00004	0.00001	0.03539	0.01802	0.0173
120	72	θ	0.5383	0.0767	0.0027	0.00127	0.6083	0.4684	0.1399
		α	0.6460	0.3445	0.0039	0.00183	0.7300	0.5620	0.1679
		β	0.0394	0.0669	0.00002	0.000018	0.0478	0.0311	0.0166
		λ	0.0115	0.1579	0.0000	0.00001	0.0180	0.0050	0.0130
		$S_{DE}(x_0)$	0.9102	0.0293	0.0011	0.0004	0.9531	0.8674	0.0857
		$h_{DE}(x_0)$	0.0270	0.1310	0.00003	0.00001	0.0345	0.0195	0.0150
		$ah_{DE}(x_0)$	0.0274	0.1325	0.00003	0.00001	0.0351	0.0197	0.0154
	96	θ	0.5312	0.0625	0.0018	0.00090	0.5902	0.4722	0.1179
		α	0.6375	0.3171	0.0027	0.00130	0.7083	0.5667	0.1415
		β	0.0375	0.0160	4.35×10^{-6}	3.98×10^{-6}	0.0415	0.0336	0.0078
		λ	0.01096	0.0965	8.95×10^{-6}	8.02×10^{-6}	0.01653	0.0054	0.0111
		$S_{DE}(x_0)$	0.9079	0.0266	0.0009	0.00042	0.9483	0.8675	0.0808
		$h_{DE}(x_0)$	0.0270	0.1290	0.00003	0.00001	0.0345	0.0196	0.0148
		$ah_{DE}(x_0)$	0.0274	0.13062	0.00003	0.00001	0.0351	0.0198	0.0152
	120	θ	0.5274	0.0548	0.0015	0.0007	0.5819	0.4729	0.1089
		α	0.6329	0.2893	0.0021	0.0011	0.6982	0.5675	0.1307
		β	0.0371	0.0037	1.84×10^{-6}	1.82×10^{-6}	0.0397	0.0344	0.0052
		λ	0.0105	0.0500	6.03×10^{-6}	5.78×10^{-6}	0.0152	0.0057	0.0094
		$S_{DE}(x_0)$	0.9065	0.0250	0.0008	0.0003	0.9455	0.8674	0.0781
		$h_{DE}(x_0)$	0.0271	0.1266	0.00002	0.00001	0.0344	0.0199	0.0145
		$ah_{DE}(x_0)$	0.0275	0.1282	0.00003	0.00001	0.0350	0.0200	0.0149

Table 3
ML averages, relative absolute biases, mean square errors, variances of the ML estimates, 95% confidence intervals of the parameters, survival, hazard rate and alternative hazard rate functions at ($x_0 = 1$) from the DE-MTLCh distribution for different samples of size *n* and censoring size r, $NR = 1000, (\theta = 0.48, \alpha = 0.6, \beta = 0.037, \lambda = 0.2)$

n	*r*	parameters	average	RAB	MSE	Variance	UL	LI	length
30	18	θ	0.6760	0.4083	0.0430	0.0045	0.8087	0.5432	0.2654
		α	0.8450	0.2459	0.0671	0.0071	1.0109	0.6790	0.3318
		β	0.0682	0.8459	0.0015	0.0006	0.1163	0.0201	0.0962
		λ	0.3402	0.7013	0.0331	0.0134	0.5676	0.1128	0.4548
		$S_{DE}(x_0)$	0.8265	0.1776	0.0168	0.0013	0.8973	0.7557	0.1416
		$h_{DE}(x_0)$	0.1051	0.1012	0.0011	0.0010	0.1695	0.0407	0.1288
		$ah_{DE}(x_0)$	0.1117	0.1139	0.0015	0.0013	0.1846	0.0388	0.1457
	24	θ	0.6184	0.2884	0.0235	0.0043	0.7475	0.4892	0.2582
		α	0.7730	0.0460	0.0367	0.0067	0.9344	0.6116	0.3228
		β	0.0495	0.3394	0.0003	0.0002	0.0783	0.0207	0.0575
		λ	0.2571	0.2859	0.0103	0.0070	0.4222	0.0921	0.3300
		$S_{DE}(x_0)$	0.8241	0.1741	0.0175	0.0025	0.9236	0.7245	0.1991
		$h_{DE}(x_0)$	0.0837	0.1226	0.0005	0.0003	0.1214	0.0461	0.0753
		$ah_{DE}(x_0)$	0.0877	0.1259	0.0006	0.0004	0.1291	0.0462	0.0828
	30	θ	0.5758	0.1997	0.0126	0.0034	0.6912	0.4605	0.2306
		α	0.7198	0.0454	0.0197	0.0054	0.8640	0.5757	0.2882
		β	0.0410	0.1085	0.00006	0.00004	0.0543	0.0276	0.0267
		λ	0.2244	0.1223	0.0047	0.0041	0.3513	0.0976	0.2537
		$S_{DE}(x_0)$	0.8048	0.1466	0.0140	0.0034	0.9207	0.6888	0.2318
		$h_{DE}(x_0)$	0.0794	0.1677	0.0004	0.00020	0.1075	0.0514	0.0560
		$ah_{DE}(x_0)$	0.0829	0.1736	0.0005	0.00024	0.1133	0.0525	0.0607
60	36	θ	0.6644	0.3842	0.0376	0.0036	0.7822	0.5461	0.2365
		α	0.8305	0.2256	0.0588	0.0056	0.9783	0.6825	0.2956
		β	0.0656	0.7737	0.0011	0.0003	0.1029	0.0283	0.0745
		λ	0.3179	0.5898	0.0226	0.0087	0.5012	0.1347	0.3665
		$S_{DE}(x_0)$	0.8258	0.1765	0.0164	0.0028	0.8898	0.7617	0.1280
		$h_{DE}(x_0)$	0.1002	0.0503	0.0005	0.0005	0.1459	0.0545	0.0914
		$ah_{DE}(x_0)$	0.1060	0.0564	0.0007	0.0006	0.1572	0.0547	0.1024
	48	θ	0.6011	0.2523	0.0187	0.0041	0.7266	0.4756	0.2510
		α	0.7514	0.0420	0.0292	0.0064	0.9083	0.5945	0.3137
		β	0.0470	0.2725	0.0002	0.0001	0.0674	0.0266	0.0408
		λ	0.2452	0.2264	0.0093	0.0072	0.4124	0.0781	0.3343
		$S_{DE}(x_0)$	0.8150	0.1611	0.0156	0.0015	0.9188	0.7111	0.2077
		$h_{DE}(x_0)$	0.0827	0.1335	0.0003	0.00022	0.1119	0.0535	0.0583
		$ah_{DE}(x_0)$	0.0864	0.1381	0.0004	0.00026	0.1183	0.0546	0.0636

Table 3 (Continued)

n	r	parameters	Average	RABs	MSE	Variance	UL	IL	Length
60	60	θ	0.5518	0.1496	0.0089	0.0038	0.6730	0.4305	0.2424
		α	0.6897	0.0418	0.0140	0.0059	0.8412	0.5382	0.3030
		β	0.0394	0.0668	0.00002	0.0000	0.0483	0.0306	0.0177
		λ	0.2122	0.0612	0.0043	0.0042	0.3397	0.0847	0.2550
		$S_{DE}(x_0)$	0.8840	0.1170	0.0108	0.0040	0.9095	0.6585	0.2509
		$h_{DE}(x_0)$	0.0814	0.1473	0.0003	0.00017	0.1074	0.0553	0.0520
		$ah_{DE}(x_0)$	0.0850	0.1527	0.0004	0.00020	0.1132	0.0568	0.0564
120	72	θ	0.6560	0.3553	0.0323	0.0032	0.7623	0.5387	0.2236
		α	0.8132	0.2004	0.0505	0.0050	0.9529	0.6734	0.2795
		β	0.0623	0.6864	0.0008	0.0002	0.0922	0.0325	0.0596
		λ	0.2957	0.4789	0.0153	0.0062	0.4504	0.1411	0.3092
		$S_{DE}(x_0)$	0.8237	0.1736	0.0159	0.0011	0.8897	0.7576	0.1320
		$h_{DE}(x_0)$	0.0959	0.0052	0.0003	0.0002	0.1299	0.0620	0.0679
		$ah_{DE}(x_0)$	0.1010	0.0074	0.0003	0.0003	0.1388	0.0632	0.0756
	96	θ	0.5756	0.2096	0.0146	0.0043	0.7125	0.4487	0.2638
		α	0.7195	0.0418	0.0229	0.0067	0.8907	0.5608	0.3298
		β	0.0444	0.2208	0.0001	0.0000	0.0606	0.0297	0.0308
		λ	0.2245	0.1442	0.0075	0.0067	0.3895	0.0681	0.3213
		$S_{DE}(x_0)$	0.8984	0.1411	0.0134	0.0035	0.9184	0.6833	0.2350
		$h_{DE}(x_0)$	0.0822	0.1339	0.0003	0.00014	0.1075	0.0578	0.0496
		$ah_{DE}(x_0)$	0.0859	0.13893	0.0003	0.00017	0.11374	0.0593	0.0540
	120	θ	0.5309	0.1060	0.0058	0.0032	0.6425	0.4179	0.2242
		α	0.6636	0.0400	0.0090	0.0050	0.8031	0.5224	0.2807
		β	0.0384	0.0386	0.0000	0.0000	0.0447	0.0318	0.0129
		λ	0.2075	0.0376	0.0040	0.0040	0.3201	0.0929	0.2271
		$S_{DE}(x_0)$	0.8621	0.0872	0.0074	0.0037	0.8840	0.6403	0.2437
		$h_{DE}(x_0)$	0.0846	0.1142	0.0002	0.00015	0.10889	0.0603	0.0485
		$ah_{DE}(x_0)$	0.0884	0.1186	0.0003	0.00018	0.1148	0.0621	0.0526

4.2 Concluding remarks

From Tables 2 and 3 one can notice that:

1. The RABs, MSEs, and variances of the ML estimates of the parameters, sf, hrf, and ahrf decrease when the sample size *n* increases. Also, the lengths of the confidence intervals get shorter when the sample size increases in most cases.

2. The RABs, MSEs and variances of the ML estimates of the parameters, sf, hrf, and ahrf decrease when the level of censoring decreases.

3. In general, all the results of the RABs, MSEs and variances obtained for complete sample sizes, are less than the corresponding results for censored samples. Also, results get better when the sample size and level of uncensored samples increases.

5. Application

This section is devoted to illustrate the flexibility and applicability of the proposed distribution using two real data sets. The proposed distribution is compared with different competitive distributions such as *discrete Marshall-Olkin inverted Topp-Leone* (DMOITL) distribution introduced by Almetwally *et al*. (2021), *discrete exponentiated Chen* (DE-Ch) distribution suggested by Alotaibi *et al*. (2023), *discrete Half Logistic* (DHL) distribution considered by Hegazy *et al*. (2020), *discrete generalized inverted exponential* (DGIE) distribution proposed by AbdElaziz *et al.* (2022), *discrete Burr* (DB) distribution presented by Krishna and Pundir (2009), and *discrete generalized Rayleigh* (DGR) distribution studied by Alamatsaz, *et al.* (2016). The fitted probability distributions are compared using *Akaike information criterion* (AIC), *Akaike information criterion with correction* (AICC), *Bayesian information criterion* (BIC) and *Hannon-Quinn information criterion* (HQIC). The best-fitting distribution corresponds to the lowest values of AIC, AICC, BIC, and HQIC, and the highest p-value associated with the *Kolmogorov-Smirnov* (K-S) goodness-of-fit test.

where $AIC = -2\ln(L) + 2k,\quad AICC = AIC + \frac{2k(k+1)}{n-k-1},\quad BIC = -2ln(L) + kln(n),$

and

$HQIC = -2\,ln\,(L) + 2k\,ln\big(ln(n)\big),$ where k is the number of the parameters, n is the sample size and L is the natural logarithm of the value of the likelihood function evaluated at the ML estimates. Tables 4 and 5 display the values of *P*-value, AIC, AICC, BIC and HQIC for the two real data sets.

Kolmogorov-Smirnov (K-S) goodness of fit test is applied to check the validity of the fitted model. The P-values are respectively 0.9434 and 0.2742. It shows that the DE-MTLCh fits the data very well.

Data set I

The first data set contains 47 observations, and it refers to numbers of daily deaths in Egypt due to COVID-19 infections from 8 March to 30 April 2020. The data are: 1, 1, 2, 2, 1, 1, 2, 4, 5, 1, 1, 3, 6, 6, 4, 1, 5, 6, 6, 8, 5, 7, 7, 9, 9, 15, 17, 11, 13, 5,14, 5, 13, 9, 19, 15, 11, 14, 12, 11, 7, 13, 10, 20, 22, 21 and 12. The data are available on worldometer website at https://www.worldometers.info/coronavirus/country/egypt/.

Table 4 presents the ML estimates, corresponding *standard errors* (SEs), P-values, AIC, AICC, BIC and HQIC. It is observed that all models fit the data set. The proposed distribution has the lowest values of the AIC, AICC, BIC and HQIC and the highest P-value. Hence, the proposed distribution is the best fit for this data compared with other distributions.

Table 4
Parameter estimates with their corresponding standard errors and information criteria for the real data set I

Models	Estimates	SEs	P-value	AIC	BIC	AICC	HQIC
DE-MTLCh	$\theta = 0.6263$ $\alpha = 1.0021$ $\beta = 0.1787$ $\lambda = 0.0466$	0.4586 0.4515 0.4670 0.4695	0.9434	293.75	301.75	309.15	302.702
DE-Ch	$\theta = 0.99$ $\alpha = 0.5$ $\beta = 0.5$	0.4517 0.4610 0.4610	0.3371	317.537	323.537	329.087	324.095
DMOITL	$\theta = 0.7883$ $\alpha = 0.9449$	0.4555 0.4526	0.0820	319.507	325.507	331.058	344.549
DGIE	$\lambda = 3.0064$ $\alpha = 0.5021$	0.4146 0.4609	0.0835	330.869	334.869	338.569	335.142
DGR	$\lambda = 0.1316$ $\alpha = 3.446$	0.4678 0.3919	0.0524	458.647	462.647	466.377	462.92
DB	$\lambda = 0.5592$ $\alpha = 0.8235$	0.4598 0.4549	0.0455	368.07	372.07	375.771	372.343

The *total time test* (TTT) plot can be used to get information about the shape of the hrf of a given data set, which helps in selecting a particular model to fit a proposed data set. The fitted pmf, PP and QQ plots indicate that DE-MTLCh distribution fit for the two real data sets.

Figures 4 shows that the TTT plot of real data set I has bathtub- shaped hrf. The fitted pmf, P-P and Q-

Q plots indicate that the DE-MTLCh distribution provides the best fit for this data.

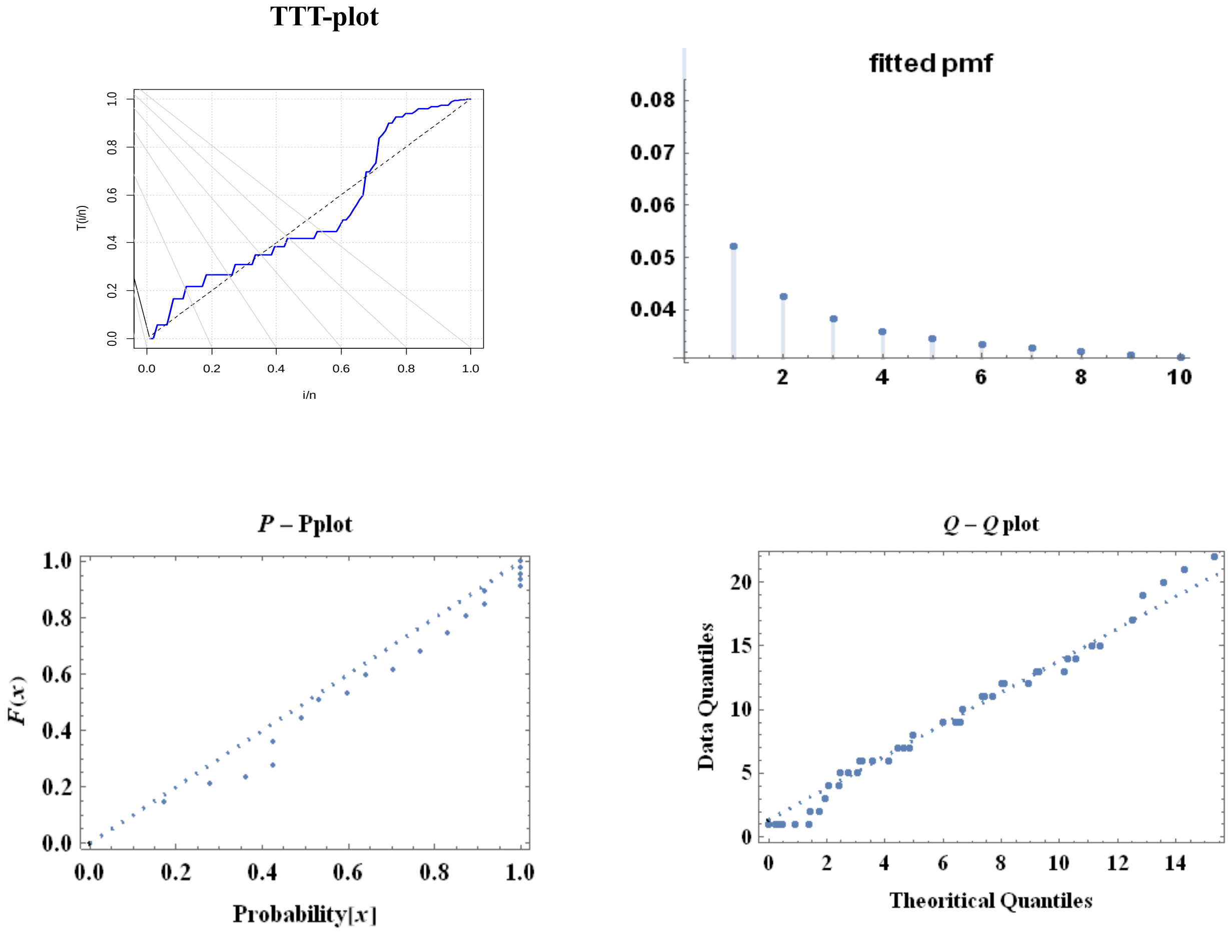

Figure 4. TTT, fitted pmf, P-P and Q-Q plots of the DE-MTLCh distribution for real Data set I

Data set II

The second data represents survival times (in weeks) for 33 patients. The patients are suffering from acute myelogenous leukemia (see Feigl and Zelen (1965)). The data are: 3, 3, 30, 3, 8, 4, 2, 4, 4, 65, 100, 108, 121, 4, 134, 16, 39, 26, 22, 1, 143, 56, 1, 5, 65, 17, 7, 16, 56, 65, 22, 43and 156.

From Table 5 one can observe that all models fit the real data set. But the proposed distribution has lowest values of the AIC, AICC, BIC and HQIC and the highest P-value. Hence, the proposed distribution is the best fit for this data compared with the other distributions.

Table 5

Parameter estimates with their corresponding standard errors
and information criteria for the real data set II

Models	Estimates	SEs	P-Value	AIC	BIC	AICC	HQIC
DE-MTLCh	$\theta = 0.4708$ $\alpha = 0.7533$ $\beta = 0.0233$ $\lambda = 0.0180$	1.3725 1.3704 1.3758 1.3759	0.2742	325.46	333.46	339.446	334.888
DE-Ch	$\theta = 0.8984$ $\alpha = 1.0544$ $\beta = 0.2098$	1.3745 1.3730 1.3757	0.0898	575.891	581.891	585.381	581.719
DMOITL	$\theta = 0.4398$ $\alpha = 0.4956$	1.3727 1.3723	0.0908	341.318	345.318	348.311	345.718
DHL	$\lambda = 16.0273$	1.2683	0.0935	345.69	347. 869	348.569	347.198
DGR	$\lambda = 0.0183$ $\alpha = 0.6904$	1.3759 1.3709	0.0929	346.248	350.248	353.241	350.648
DB	$\lambda = 0.4224$ $\alpha = 7494$	1.3729 1.3705	0.0810	343.508	347.508	350.501	347.908

Figures 5 shows that the TTT plot of the real data set II has bathtub-shaped hazard rate. The fitted pmf, P-P and Q-Q plots indicate that the DE-MTLCh distribution fits the data very well.

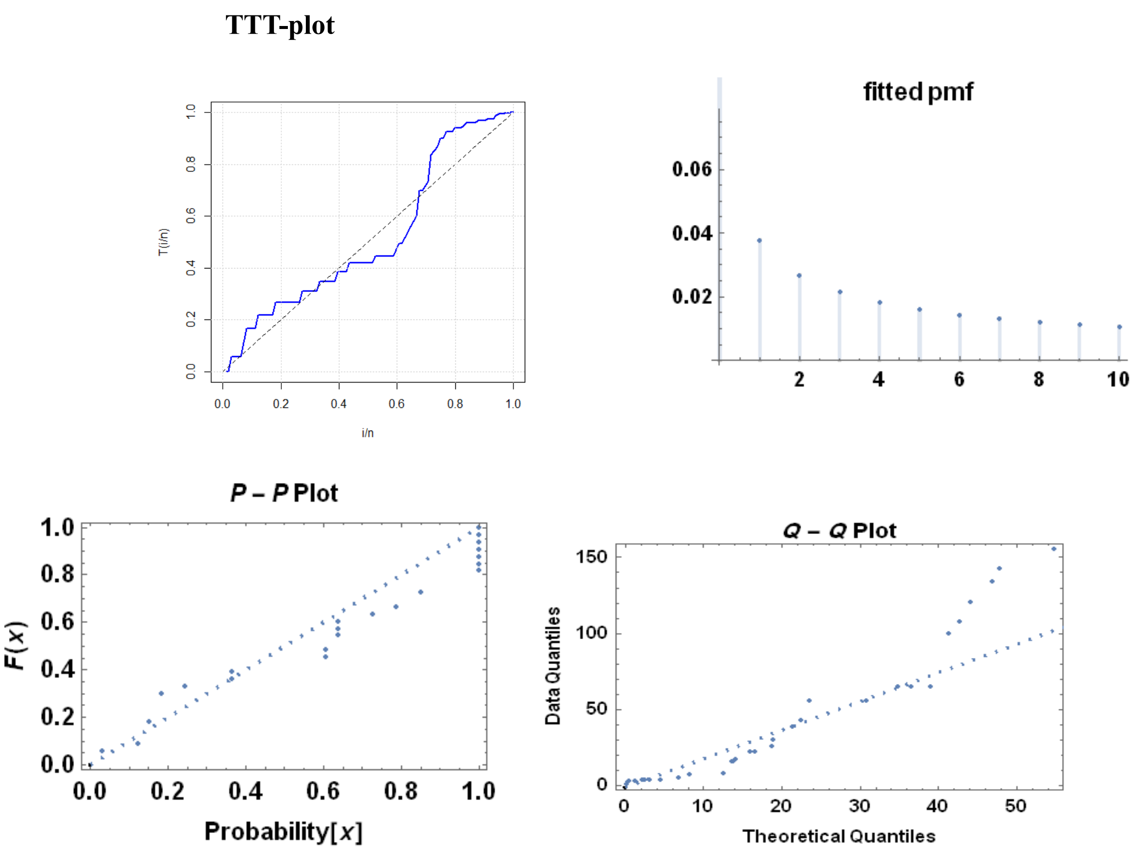

Figure 5. TTT-plot, fitted pmf, PP-plot, and Q-Q plot of the DE-MTLCh distribution for the Data set II

6. Conclusion

In this paper, a new distribution with four parameters named DE-MTLCh distribution is introduced. The pmf of the proposed distribution has unimodal, decreasing, and increasing curves. Also, the corresponding hrf can be decreasing, increasing and bathtub shapes. Some important statistical properties are obtained such as quantile, moments, order statistics and Rényi Entropy. Numerical values for the mean, variance, Sk, Kur, the ID and Cv are presented. The method of the ML is used to estimate the unknown parameters, sf, hrf and ahrf. Finally, the flexibility and applicability of the DE-MTLCh distribution in real life was illustrated by applying two real data sets. The DE-MTLCh distribution is more suitable for modeling real data sets as it is a better alternative to some other distributions.

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توب ليون شن المعدل الاسي المتقطع وتطبيقات لتوزيع الخصائص الاحصائية

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المستخلص:

تناول هذا البحث تقديم توزيع جديد بأربعة معلمات والذي يسمى توزيع توب ليون شن المعدل الاسي المتقطع، وذلك من خلال استخدام المدخل العام للتحويل من توزيعات مستمرة إلى توزيعات متقطعة. كما تناول أيضا أهمية تحويل توزيعات الحياة المستمرة إلى توزيعات متقطعة. وتم إيضاح أهمية التوزيع المقترح حيث يمكن تطبيقه في العديد من المجالات المختلفة. كما تم الحصول على بعض الخصائص الإحصائية لهذا التوزيع مثل الدالة الربيعية، متوسط العمر المتبقي، متوسط الوقت حتى الفشل، متوسط الوقت بين فشلين،الانتروبي، والعزوم والإحصاءات الترتيبية. تم تقدير معالم هذا التوزيع باستخدام طريقة الامكان الاكبرمعتمداًعلي عينات خاضعة للرقابة من النوع الثاني ومن ثم إيجاد مقدرات دوال البقاء، معدل الفشل ومعدل الفشل البديل من خلال خاصية الثبات أو عدم الاختلاف التي تتميز بها مقدرات الإمكان الأكبر وايضاً تم استخدام التوزيع التقريبي لمقدرات الإمكان الأكبر وذلك لتكوين فترات الثقة لكلا من معالم التوزيع ودالة البقاء ، دالة معدل الفشل ودالة معدل الفشل البديل. وقد تم استخدام أسلوب المحاكاة مونت كارلو لتوضيح النتائج النظرية التي تم التوصل إليها وذلك من خلال قياس جودة هذه المقدرات. كما تم ايضاً استخدام بيانات حقيقية لإيضاح مرونة وقابلية تطبيق التوزيع المقترح في تطبيقات الحياة العملية.

الكلمات المفتاحية:

طريقة دالة البقاء - توزيع توب ليون شن المعدل – فترات الثقة - طريقة الامكان الاكبر – طريقة محاكاة سلسلة ماركوف مونت كارلو.