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Research article

Bayesian and Maximum Likelihood Estimation for Mixture Models of the New Topp-Leone-G Family

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Abstract: This paper introduces a new family of continuous distributions, the mixture of two components from the new Topp-Leone-G family. Statistical properties of the proposed family are explored, with a focus on the mixture of two new Topp-Leone exponential distributions as a sub-model. The study uses maximum likelihood and Bayesian methods under Type-II censoring to estimate the unknown parameters, reliability, and hazard rate functions. A simulation study evaluates the performance of the estimators. Finally, two real data sets are utilized to validate the simulated results and demonstrate the practical applicability of the proposed distribution in real life.

Keywords: Mixture distribution, New Topp-Leone family, Exponential distribution, Maximum likelihood estimation, Bayesian estimation.

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1. Introduction:

Mixture models, which are a convex combination of two or more probability density functions, are applied to model population heterogeneity, generalize distributional assumptions, and perform tasks such as clustering and classification. Mixture distributions have been successfully used in various fields, including astronomy, biology, genetics, medicine, psychiatry, economics, engineering, and marketing, as well as many other areas in the biological, physical, and social sciences.

Several studies on finite mixture models when the component belongs to the same family were presented by many authors. Ahmed et al. (1997) obtained approximate Bayes estimators for the parameters of the mixture of two Weibull distributions under Type-II censoring. AL-Hussaini et al.

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(2000) concerned with the statistical properties of the finite mixture of two Gompertz lifetime models. Jaheen (2005) adopted both the maximum likelihood (ML) and Bayesian approaches to discuss the problem of estimating the parameters using the finite mixture of two exponential distributions based on record statistics. Mahmoud and Ghazal (2012) succeeded in characterizing a finite mixture of two components of exponentiated family of distributions based on recurrence relations for moment and conditional moment generating functions of generalized order statistics. Kharazmi et al. (2022) considered the 2-component mixture of Topp-Leone distribution and obtained classical, Bayes estimators based on the complete sample. Recently, Crisci et al. (2023) provide a method based on quantiles to estimate the parameters of a finite mixture of Fréchet distributions, for a large sample of strongly dependent data.

A finite mixture between two distributions or more can be defined by the following *probability density function* (pdf) and *cumulative distribution function* (cdf)

$$f(x) = \sum_{i=1}^{k} p_i g_i(x), \quad (1)$$

and

$$F(x) = \sum_{i=1}^{k} p_i \, G_i(x), \quad (2)$$

where, $p_i{'}s$ are non-negative quantities that sum to one; where

$0 \leq p_i \leq 1$, $(i = 1, \cdots, k)$, and $\sum_{i=1}^{k} p_i = 1$.

The quantities p_i are called the mixing proportions or weights, also, the $g_i(x)$ in (1) and $G_i(x)$ in (2) are called the i^{th} components of the mixture. [See Mclachlan and Peel (2000)].

Several families of distributions are constructed by adding one or more parameters to a distribution function to generate flexible continuous distributions. Some of these families of distributions are: Marshall-Olkin generated (MO-G) family by Marshall and Olkin (1997), beta-G by Eugene et al. (2002), Kumaraswamy-G (Kw-G) by Cordeiro and Castro (2011), transformed transformer (T-X) family by Alzaatreh et al. (2013), exponentiated T-X by Alzaghal et al. (2013), Weibull-G by Bourguignon et al.(2014), exponentiated half-logistic by Cordeiro et al. (2014a), Lomax-G by Cordeiro et al. (2014b), Zografos-Balakrishnan-G by Nadarajah et al. (2015) and, Lindley-G by Ozel and Cakmakyapan (2017).

Hassan et al. (2021) introduced a new generating lifetime family called the new Topp-Leone generating (NTL-G) family as an alternative to Topp-Leone-G (TL-G) family. The advantage of this family is its ability to provide better fitting results compared to other distribution families.

The cdf and pdf of the NTL-G family are defined by

$$F(x;\alpha) = \left(1 - e^{-2H(x)}\right)^{\alpha} , \quad x \in \mathbb{R}, \quad (3)$$

and

$$f(x;\alpha) = \frac{2\alpha g(x)}{[1-G(x)]^2} e^{-2H(x)}\left(1-e^{-2H(x)}\right)^{\alpha-1}, \tag{4}$$

where, $\left[1-e^{-2H(x)}\right] \in [0,1]$, and $H(x) = \frac{G(x)}{1-G(x)}$ is the odds ratio.

The motivation for this study is to generalize mixture distributions by introducing the mixing of two components from the same family, thereby generating a variety of new mixture distributions.

This paper is organized as follows: In Section 2, the mixture of two components of new Topp-Leone-G (MNTL-G) families with some properties are discussed. The mixture of two NTL-exponential (MNTL-Ex) distribution is presented as a sub-model from the MNTL-G families in Section 3, some statistical properties, ML estimation and Bayesian estimation for the unknown parameters based on Type-II censoring using squared error (SE) and linear exponential (LINEX) loss functions, are obtained. In Section 4, a simulation study is conducted to assess the performance of the ML and Bayes estimators of the parameters of the MNTL-Ex distribution. Finally, in Section 5, two real data sets are used to validate the simulation results.

2. Mixture of Two Components of New Topp Leon-G Family

If the cdf and pdf of NTL-G family are given by (3) and (4), then the mixture of two NTL-G (MNTL-G) families can be obtained as given below. The pdf and cdf of MNTL-G family, respectively, are

$$f\left(x;\underline{\psi}\right) = \frac{2p\alpha_1 g_1(x)}{[1-G_1(x)]^2} e^{-2H_1(x)}\left[1-e^{-2H_1(x)}\right]^{\alpha_1-1} + \frac{2(1-p)\alpha_2 g_2(x)}{[1-G_2(x)]^2} e^{-2H_2(x)}\left[1-e^{-2H_2(x)}\right]^{\alpha_2-1}, \quad x \in \mathbb{R} \tag{5}$$

where $H_k\left(x;\underline{\psi}\right) = \frac{G_k(x)}{1-G_k(x)}, \quad k = 1,2.$

and

$$F\left(x;\underline{\psi}\right) = p\left[1-e^{-2H_1(x)}\right]^{\alpha_1} + (1-p)\left[1-e^{-2H_2(x)}\right]^{\alpha_2}. \tag{6}$$

The corresponding rf, hrf and rhrf, respectively, are

$$S\left(x;\underline{\psi}\right) = 1 - p\left[1-e^{-2H_1(x)}\right]^{\alpha_1} - (1-p)\left[1-e^{-2H_2(x)}\right]^{\alpha_2}, \tag{7}$$

$$h\left(x;\underline{\psi}\right) = \frac{\frac{2p\alpha_1 g_1(x)}{[1-G_1(x)]^2} e^{-2H_1(x)}\left[1-e^{-2H_1(x)}\right]^{\alpha_1-1} + \frac{2(1-p)\alpha_2 g_2(x)}{[1-G_2(x)]^2} e^{-2H_2(x)}\left[1-e^{-2H_2(x)}\right]^{\alpha_2-1}}{1-p[1-e^{-2H_1(x)}]^{\alpha_1} - (1-p)[1-e^{-2H_2(x)}]^{\alpha_2}}, \tag{8}$$

and

$$rh\left(x;\underline{\psi}\right) = \frac{\frac{2p\alpha_1 g_1(x)}{[1-G_1(x)]^2}e^{-2H_1(x)}\left[1-e^{-2H_1(x)}\right]^{\alpha_1-1}+\frac{2(1-p)\alpha_2 g_2(x)}{[1-G_2(x)]^2}e^{-2H_2(x)}\left[1-e^{-2H_2(x)}\right]^{\alpha_2-1}}{p[1-e^{-2H_1(x)}]^{\alpha_1}+(1-p)[1-e^{-2H_2(x)}]^{\alpha_2}}. \tag{9}$$

2.1 Some properties of the mixture of two new Topp Leone-G families

In this section, some statistical properties of the MNTL-G families are derived.

2.1.1 Quantile function

The quantile function of MNTL-G families can be derived from (6), by solving $Q(u) = F^{-1}(u)$, *i.e.*, $F(x) = u$ for x , as follows:

$$p\left[1-e^{-2H_1(x)}\right]^{\alpha_1}+(1-p)\left[1-e^{-2H_2(x)}\right]^{\alpha_2}-u=0,\ \ u\in[0,1]. \tag{10}$$

Equation (10) can be solved numerically to obtain the u^{th} quantile. Then, from this expression, one can derive the quantiles: $Q_2 = Q(u = 0.5)$ to obtain the median, $Q_1 = Q(u = 0.25)$ and $Q_3 = Q(u = 0.75)$ to obtain the first and third quantiles.

2.1.2 Moments

The moments can be used to determine the features and shapes of a distribution such as, dispersion, spread and symmetry, which can be measured, by mean, variance, kurtosis and skewness respectively. Let X be a random variable with the density given in (5). Then the ordinary r^{th} moment of MNTL-G family, say μ_r' is given by

$$\mu_r' = E(x^r) = \int_0^\infty x^r f(x)dx , \tag{11}$$

where $f(x) = pf_1(x) + (1-p)f_2(x),$
and

$$f_k(x) = 2\alpha_k g_k(x)[1-G_k(x)]^{-2}e^{-2H_k(x)}\left[1-e^{-2H_k(x)}\right]^{\alpha_k-1}, k = 1,2 . \tag{12}$$

Using the generalized binomial expansion for $n > 0$, n is real non integer

$(1-x)^n = \sum_{i=0}^\infty(-1)^i\binom{n}{i}x^i, \quad |x| < 1$,

the Taylor expansion $e^x = \sum_{j=0}^\infty \frac{x^j}{j!}$,

and the sum of the infinite series $(1-x)^{-n} = \sum_{s=0}^\infty\binom{n+s+1}{s}x^s , |x| < 1.$

Then, (12) can rewritten as follows:

$$f_k(x) = \sum_{i_k,j_k,s_k=0}^{\infty} (-1)^{i_k+j_k}(2)^{j_k+1}\frac{(i_k+1)^{j_k}}{(j_k)!}\binom{j_k+s_k+1}{s_k}\binom{\alpha_k-1}{i_k}\alpha_k g_k(x)[G_k(x)]^{j_k+s_k}. \tag{13}$$

Therefore, the r^{th} moment of MNTL-G family is obtained as given below

$$\mu_r' = \sum_{i_1,j_1,s_1=0}^{\infty} \varphi_{i_1,j_1,s_1}\omega_{j_1,s_1,r} + \sum_{i_2,j_2,s_2=0}^{\infty} \varphi_{i_2,j_2,s_2}\omega_{j_2,s_2,r}\ , \tag{14}$$

where $\varphi_{i_k,j_k,s_k} = (-1)^{i_k+j_k}(2)^{j_k+1}\frac{(i_k+1)^{j_k}}{(j_k)!}\binom{j_k+s_k+1}{s_k}\binom{\alpha_k-1}{i_k},\ k = 1,2,$

$$\omega_{j_1,s_1,r} = p\alpha_1\int_0^{\infty} x^r g_1(x)[G_1(x)]^{j_1+s_1}dx,$$

and

$$\omega_{j_2,s_2,r} = (1-p)\alpha_2\int_0^{\infty} x^r g_2(x)[G_2(x)]^{j_2+s_2}dx.$$

Moment generating function

The moment generating function denoted by $M_x(t)$ of the MNTL-G family is

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}f(x)dx$$

$$= p\alpha_1\,\varphi_{i_1,j_1,s_1}\int_0^{\infty} e^{tx}g_1(x)[G_1(x)]^{j_1+s_1}dx + (1-p)\alpha_2\,\varphi_{i_2,j_2,s_2}\int_0^{\infty} e^{tx}g_2(x)[G_2(x)]^{j_2+s_2}dx,$$

Then, using the Taylor expansion, the moment generating function is obtained based on the moments as

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'. \tag{15}$$

2.1.3 Distribution of order statistics

Considering a random sample from the MNTL-G family, $X_1, X_2, \dots, X_n$. Let $X_{(1)} \le X_{(2)} \le \dots \le X_{(n)}$ be the corresponding order statistics, then pdf of the i^{th} order statistic, $X_{i:n}$, is given by

$$f_{i:n}\left(x;\underline{\psi}\right) = \frac{n!}{(n-i)!\,(i-1)!}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}f\left(x;\underline{\psi}\right)\left[F\left(x;\underline{\psi}\right)\right]^{i+r-1}, x_{(i)} > 0.$$

Substituting (5) and (6) in the previous equation, one obtains the pdf of the i^{th} order statistics of MNTL-G distribution

$$f_{i:n}\left(x;\underline{\psi}\right)$$

$$=\frac{n!}{(n-i)!\,(i-1)!}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}\left[\frac{2p\alpha_1 g_1\left(x;\underline{\psi}\right)}{\left[1-G_1\left(x;\underline{\psi}\right)\right]^2}e^{-2H_1\left(x;\underline{\psi}\right)}\left[1-e^{-2H_1\left(x;\underline{\psi}\right)}\right]^{\alpha_1-1}\right.$$

$$\left.+\frac{2(1-p)\alpha_2 g_2\left(x;\underline{\psi}\right)}{\left[1-G_2\left(x;\underline{\psi}\right)\right]^2}e^{-2H_2(x)}\left[1-e^{-2H_2\left(x;\underline{\psi}\right)}\right]^{\alpha_2-1}\right]\left[p\left[1-e^{-2H_1\left(x;\underline{\psi}\right)}\right]^{\alpha_1}\right.$$

$$\left.+(1-p)\left[1-e^{-2H_2\left(x;\underline{\psi}\right)}\right]^{\alpha_2}\right]^{i+r-1},\qquad x_{(i)}>0\,,\tag{16}$$

where $;\underline{\psi}$ is the parameter vector.

The smallest and largest order statistics can be obtained by substituting $i=1$ and n, respectively, in (16).

3. Mixture of Two New Topp Leone-Exponential Distribution

In this section, the mixture of two NTL-exponential (MNTL-Ex) distributions is proposed as a sub-model from the mixture of two NTL families. Some statistical properties are studied, ML and Bayesian estimation of the unknown parameters of the MNTL-Ex are obtained.

3.1 Description of the distribution

The mixture of two NTL-exponential (MNTL-Ex) distributions can be obtained when $g_i(x)=\lambda_i e^{-\lambda_i x}$ and $G_i(x)=1-e^{-\lambda_i x}$, $i=1,2$. Then, the pdf of the MNTL-Ex distribution is

$$f_M(x;\,\underline{\theta})=2p\alpha_1\lambda_1 e^{\lambda_1 x-2\left(e^{\lambda_1 x}-1\right)}\left[1-e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1-1}$$

$$+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2\left(e^{\lambda_2 x}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2-1},\underline{\theta}>\underline{0}\,;\ x>0,\tag{17}$$

where $\underline{\theta}=(\alpha_1,\lambda_1,\alpha_2,\lambda_2,p\ )$.

The corresponding cdf, rf, hrf and rhrf of the MNTL-Ex distribution, respectively, are

$$F_M(x;\underline{\theta})=p\left[1-e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1}+(1-p)\left[1-e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2},\underline{\theta}>\underline{0}\,;x>0,\tag{18}$$

$$S_M(x;\underline{\theta})=1-p\left[1-e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2},\underline{\theta}>\underline{0}\,;\ x>0,\tag{19}$$

$$h_M(x;\underline{\theta})$$

$$=\frac{2p\alpha_1\lambda_1 e^{\lambda_1 x-2\left(e^{\lambda_1 x}-1\right)}\left[1-e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1-1}+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2\left(e^{\lambda_2 x}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2-1}}{1-p\left[1-e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2}},$$

$$\underline{\theta} > \underline{0}\,;\ x > 0, \qquad (20)$$

and

$rh_M(x;\underline{\theta})$

$$= \frac{2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}}{p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} + (1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}}.$$

$$\underline{\theta} > \underline{0}\,;\ x > 0, (21)$$

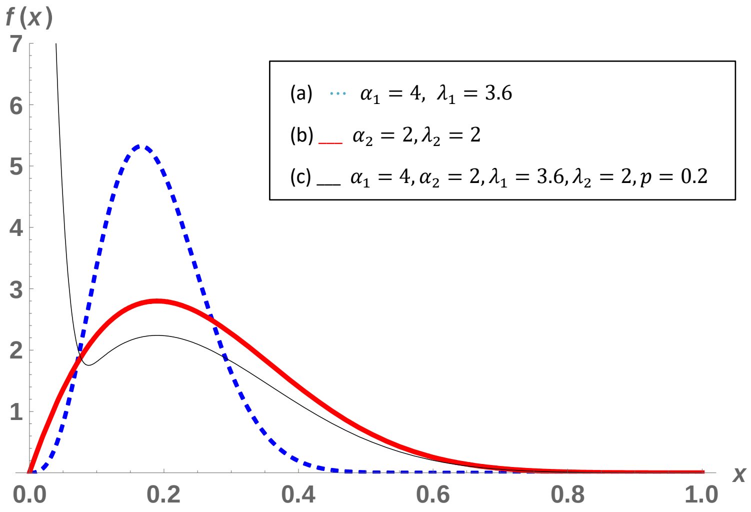

Figure 1. Plots of the two components and their mixture of the MNTL-Ex distribution at different values of the parameters

Figure 1 shows the pdf of the first component with (α_1, λ_1) in (a), the second component with (α_2, λ_2) in plot (b) and their mixture MNTL-Ex with the parameters $\alpha_1, \lambda_1, \alpha_2, \lambda_2, p$ in plot (c).

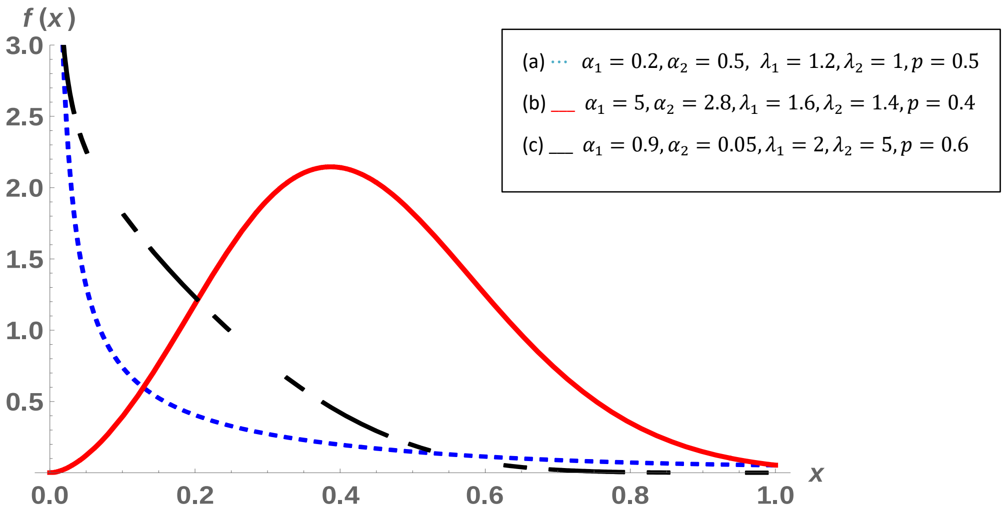

Figure 2. Plots of the pdf of the MNTL-Ex distribution at different values of the parameters

Figure 2 displays different shapes of the pdf for the MNTL-Ex distribution. The densities in (a) and (c) are decreasing. However, the density in (b) is unimodal and symmetric.

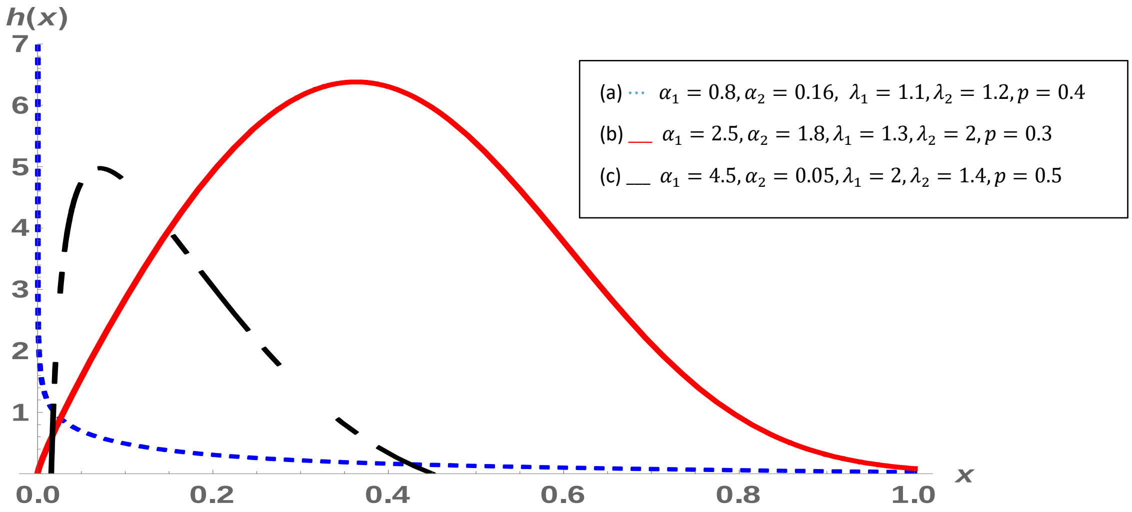

Figure 3. Plots of the hrf of the MNTL-Ex distribution at different values of the parameters

Figure 3 presents different shapes of the hrf for the MNTL-Ex distribution. The hrf in (a) is decreasing. The hrf in (b) and (c) are unimodal and right skewed.

3.2 Some properties of the proposed distribution

3.2.1 Quantile function

The quantile function of the MNTL-Ex distribution can be obtained by substituting $H_k(x) = \frac{G_k(x)}{1-G_k(x)}, k = 1,2,$ where $G_k(x) = 1 - e^{-\lambda_i x}, i = 1,2$ into (10). Thus, $H_k(x) = e^{\lambda_i x}(1 - e^{-\lambda_i x})$ and solving the following equation

$$p\left[1 - e^{-2e^{\lambda_1 x}(1-e^{-\lambda_1 x})}\right]^{\alpha_1} + (1-p)\left[1 - e^{-2e^{\lambda_2 x}(1-e^{-\lambda_2 x})}\right]^{\alpha_2} - u = 0. \quad (22)$$

Also, a random sample from the MNTL-Ex distribution can be generated using Uniform distribution in (22).

3.2.2 Moments

Let $X \sim \text{MNTL} - \text{Ex}\left(x, p_j, \alpha_j, \lambda_j\right)$, then the r^{th} moment of the MNTL-Ex distribution is given by

$$\mu'_r = E(x^r) = \sum_{j=1}^{2} p_j E_j(x^r) = \sum_{j=1}^{2} p_j \int_0^\infty x^r f(x)\, dx$$

$$= \sum_{j=1}^{2} 2\alpha_j \lambda_j p_j \int_0^\infty x^r e^{\lambda_j x - 2\left(e^{\lambda_j x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_j x}-1\right)}\right]^{\alpha_j - 1} dx. \quad (23)$$

Let $\mu'_r = \mathbf{I}_1 + \mathbf{I}_2$ where, $\mathbf{I}_1$ and $\mathbf{I}_2$ are obtained as follows

$$\mathbf{I}_1 = 2\alpha_1\lambda_1 p \int_0^{\infty} x^r e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} dx,$$

and

$$\mathbf{I}_2 = 2\alpha_2\lambda_2(1-p) \int_0^{\infty} x^r e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} dx.$$

Using Binomial expansion, then $\mathbf{I}_1$ can be written as

$$\mathbf{I}_1 = 2\alpha_1\lambda_1 p \textstyle\sum_{i=0}^{\infty} (-1)^i \binom{\alpha_1 - 1}{i} \int_0^{\infty} x^r e^{\lambda_1 x}\, e^{-2(i+1)(e^{\lambda_1 x}-1)} dx.$$

Using Taylor expansion, then

$$\mathbf{I}_1 = 2\alpha_1\lambda_1 p \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\lambda_1{}^j}{j!} (-1)^i \binom{\alpha_1 - 1}{i} \int_0^{\infty} x^{(r+j)}\, e^{-2(i+1)(e^{\lambda_1 x}-1)} dx$$

$$= 2\alpha_1 p(\lambda_1)^{j+1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i}{j!} \binom{\alpha_1 - 1}{i} \int_0^{\infty} x^{(r+j)}\, e^{2(i+1)} e^{-2(i+1)e^{\lambda_1 x}} dx$$

$$= \sum_{i,j,k=0}^{\infty} (2)^{k+1} (-1)^{i+k} \frac{(i+1)^k}{j!\,k!} \alpha_1 p(\lambda_1)^{j+1} \binom{\alpha_1 - 1}{i} e^{2(i+1)} \int_0^{\infty} x^{(r+j)}\, e^{k e^{\lambda_1 x}} dx.$$

Similarly, $\mathbf{I}_2$ can be obtained and thus the moments are obtained in non-explicit form, so it must be solved numerically.

3.2.3 Distribution of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the MNTL-Ex distribution. Then by substituting with $g_i(x) = \lambda_i e^{-\lambda_i x}$, $G_i(x) = \left(1 - e^{-\lambda_i x}\right)$ and $H_i(x) = \left(e^{-\lambda_i x} - 1\right)$ in (16), the pdf of the i th order statistic, $X_{i:n}$,can be obtained as follows:

$$f_{i:n}(x;\underline{\theta}) = \frac{n!}{(n-i)!\,(i-1)!} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} \Bigg[2p\alpha_1\lambda_1 e^{\lambda_1 x} e^{-2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1}$$

$$+ 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x} e^{-2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1}\Bigg]$$

$$\times \left[p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} + (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right]^{i+r-1}, \quad x_{(k)} > 0, \qquad (24)$$

where $\underline{\theta} = (\alpha_1, \lambda_1, \alpha_2, \lambda_2, p\ )$.

The smallest and largest order statistics of the MNTL-Ex distribution can be obtained as a special case by substituting $i = 1, n$, respectively, in (24).

3.2.4 Entropy

The concept of entropy is a useful tool to measure the variation of uncertainty, randomness or variation of a random variable. There are many different entropy measures discussed in the previous literature, with the most flexible ones in certain areas being Tsallis (1988), Shannon (1948), and Rényi (1961) entropies. These entropies are used in many applications in various fields such as physics, ecology, engineering, chemistry and economics.

i. Tsallis entropy

Tsallis entropy of a random variable X with the pdf (17) of the MNTL-Ex distribution, denoted by $H_s(x)$, is defined by

$$H_s(x) = \frac{1}{(s-1)}\left(1 - \int_0^\infty \left(f_M(x;\underline{\theta})\right)^s dx\right), \quad s > 0, s \neq 1, \tag{25}$$

$$H_s(x) = \frac{1}{(s-1)}\left(1 - \int_0^\infty \left(2p_i\alpha_i\lambda_i e^{\lambda_i x - 2\left(e^{\lambda_i x - 1}\right)}\left[1 - e^{-2\left(e^{\lambda_i x - 1}\right)}\right]^{\alpha_i - 1}\right)^s dx\right), \tag{26}$$

where $i = 1,2$ and $p_2 = 1 - p_1$.

ii. Shannon entropy

Let x be a random variable with pdf of the MNTL-Ex distribution, the Shannon entropy, denoted by $H(x)$, is defined by

$$H(x) = -E(\ln f_M(x)) = -\int_0^\infty \ln\left(f_M(x;\underline{\theta})\right) f_M(x;\underline{\theta})dx \tag{27}$$

$$= -\int_0^\infty \begin{pmatrix} \ln\left(2p_i\alpha_i\lambda_i e^{\lambda_i x - 2\left(e^{\lambda_i x - 1}\right)}\left[1 - e^{-2\left(e^{\lambda_i x - 1}\right)}\right]^{\alpha_i - 1}\right) \\ \times \left(2p_i\alpha_i\lambda_i e^{\lambda_i x - 2\left(e^{\lambda_i x - 1}\right)}\left[1 - e^{-2\left(e^{\lambda_i x - 1}\right)}\right]^{\alpha_i - 1}\right) dx. \end{pmatrix} \tag{28}$$

Note that: The Tsallis entropy is a generalization of Shannon entropy, as Shannon entropy can be obtained from the Tsallis entropy when $s \to 1$.

iii. Rényi entropy

The Rényi entropy of the nonnegative random variable X is

$$R_\delta(x) = \frac{1}{(1-\delta)} ln\left(\int_0^\infty \left(f(x;\underline{\theta})\right)^\delta dx\right), \quad \delta > 0, \delta \neq 1. \tag{29}$$

Then, the Rényi entropy of the MNTL-Ex distribution can be obtained as

$$R_\delta(x) = \frac{1}{(1-\delta)} ln \int_0^\infty \left(2p_i\alpha_i\lambda_i e^{\lambda_i x - 2\left(e^{\lambda_i x - 1}\right)}\left[1 - e^{-2\left(e^{\lambda_i x - 1}\right)}\right]^{\alpha_i - 1}\right)^\delta dx. \tag{30}$$

As $\delta \to 1$, Rényi entropy tends to Shannon entropy.

4. Estimation for Mixture of the Two New Topp-Leone-Exponential Distribution

This section introduces the ML and Bayesian estimations for the unknown parameters, rf and hrf of the MNTL-Ex distribution based on Type II censoring.

4.1 Maximum likelihood estimation

4.1.1 Point estimation

Considering that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$ is a sample of size r from the MNTL-Ex distribution with the vector of parameters $\underline{\theta} = (p_i, \alpha_i, \lambda_i, \beta_i)$, then the likelihood function is given by

$$L_M(\theta; \underline{x}) = \left\{\prod_{i=1}^{r} f(x_i; \underline{\theta})\right\}\left(s(x_r; \underline{\theta})\right)^{n-r}. \tag{31}$$

By substituting (17) and (19) in (31), the likelihood function of the MNTL-Ex distribution

$$\begin{aligned} L_M(\underline{\theta}; \underline{x}) = &\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right. \\ &\left.\left.+\,2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\} \\ &\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r}. \end{aligned} \tag{32}$$

The natural logarithm of the likelihood function is given by

$$\begin{aligned} \ell_M \equiv lnL_M(\underline{\theta}; \underline{x}) & \\ = &\sum_{i=1}^{r} ln\left\{2p\alpha_1\lambda_1 e^{\lambda_1 x_i-2(e^{\lambda_1 x_i}-1)}\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1-1}\right. \\ &\left.+\,2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x_i-2(e^{\lambda_2 x_i}-1)}\left[1-e^{-2(e^{\lambda_2 x_i}-1)}\right]^{\alpha_2-1}\right\} \\ &+(n-r)ln\left[1-p\left[1-e^{-2(e^{\lambda_1 x_r}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x_r}-1)}\right]^{\alpha_2}\right]. \end{aligned} \tag{33}$$

Differentiation the log likelihood function in (33) with respect to $\alpha_1, \lambda_1, \alpha_2, \lambda_2$ and p, respectively, one gets

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha_1} = &\sum_{i=1}^{r}\frac{2p\lambda_1 e^{\lambda_1 x_i-2(e^{\lambda_1 x_i}-1)}\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1-1}\left[1+\alpha_1 ln\left(1-e^{-2(e^{\lambda_1 x_i}-1)}\right)\right]}{A_1(x_i, \underline{\theta})} \\ &-(n-r)\frac{p\left(\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1} ln\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]\right)}{A_2(x_r, \underline{\theta})}, \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\lambda_1} = &\sum_{i=1}^{r}\left\{\begin{array}{c}\dfrac{\left(4p\alpha_1\lambda_1(\alpha_1-1)x_i e^{2\lambda_1 x_i-4(e^{\lambda_1 x_i}-1)}\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1}\right)}{A_1(x_i, \underline{\theta})} \\ +\dfrac{\left(2p\alpha_1 e^{\lambda_1 x_i-2(e^{\lambda_1 x_i}-1)}\left[1+\lambda_1 x_i(1-2e^{\lambda_1 x_i})\right]\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1-1}\right)}{A_1(x_i, \underline{\theta})}\end{array}\right\} \\ &-2(n-r)\frac{p\alpha_1 x_i e^{\lambda_1 x_i-2(e^{\lambda_1 x_i}-1)}\left[1-e^{-2(e^{\lambda_1 x_i}-1)}\right]^{\alpha_1-1}}{A_2(x_r, \underline{\theta})}, \end{aligned} \tag{35}$$

$$\frac{\partial \ell}{\partial \alpha_2}=\sum_{i=1}^{r}\frac{2(1-p)\lambda_2 e^{\lambda_2 x_i-2\left(e^{\lambda_2 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2-1}\left[1+\alpha_2 ln\left(1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right)\right]}{A_1\left(x_i,\underline{\theta}\right)}$$

$$-(n-r)\frac{(p-1)\left(\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2} ln\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]\right)}{A_2}, \tag{36}$$

$$\frac{\partial \ell}{\partial \lambda_2}=\sum_{i=1}^{r}\left\{\begin{array}{c}\dfrac{\left(4(1-p)\alpha_2\lambda_2(\alpha_2-1)x_i e^{2\lambda_2 x_i-4\left(e^{\lambda_2 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2}\right)}{A_1\left(x_i,\underline{\theta}\right)}\\ +\dfrac{\left(2(1-p)\alpha_2 e^{\lambda_2 x_i-2\left(e^{\lambda_2 x_i}-1\right)}\left[1+\lambda_2 x_i\left(1-2e^{\lambda_2 x_i}\right)\right]\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2-1}\right)}{A_1\left(x_i,\underline{\theta}\right)}\end{array}\right\}$$

$$-2(n-r)\frac{(1-p)\alpha_2 x_i e^{\lambda_2 x_i-2\left(e^{\lambda_2 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2-1}}{A_2(x_r,\underline{\theta})}, \tag{37}$$

and

$$\frac{\partial \ell}{\partial p}=\sum_{i=1}^{r}\frac{f_1\left(x_i,\underline{\theta}\right)-f_2\left(x_i,\underline{\theta}\right)}{A_1(x_i,\underline{\theta})}$$

$$+(n-r)\frac{\left[1-e^{-2\left(e^{\lambda_2 x_r}-1\right)}\right]^{\alpha_2}-\left[1-e^{-2\left(e^{\lambda_1 x_r}-1\right)}\right]^{\alpha_1}}{A_2(x_r,\underline{\theta})}, \tag{38}$$

where $A_1(x_i,\underline{\theta})$, $A_2(x_r,\underline{\theta})$, $f_1\left(x_i,\underline{\theta}\right)$ and $f_2\left(x_i,\underline{\theta}\right)$ are

$$A_1\left(x_i,\underline{\theta}\right)=\left\{\begin{array}{c}2p\alpha_1\lambda_1 e^{\lambda_1 x_i-2\left(e^{\lambda_1 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_1 x_i}-1\right)}\right]^{\alpha_1-1}\\ +2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x_i-2\left(e^{\lambda_2 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2-1}\end{array}\right\}, \tag{39}$$

$$A_2\left(x_r,\underline{\theta}\right)=\left[1-p\left[1-e^{-2\left(e^{\lambda_1 x_r}-1\right)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2\left(e^{\lambda_2 x_r}-1\right)}\right]^{\alpha_2}\right], \tag{40}$$

$$f_1\left(x_i,\underline{\theta}\right)=2\alpha_1\lambda_1 e^{\lambda_1 x_i-2\left(e^{\lambda_1 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_1 x_i}-1\right)}\right]^{\alpha_1-1}, \tag{41}$$

and

$$f_2\left(x_i,\underline{\theta}\right)=2\alpha_2\lambda_2 e^{\lambda_2 x_i-2\left(e^{\lambda_2 x_i}-1\right)}\left[1-e^{-2\left(e^{\lambda_2 x_i}-1\right)}\right]^{\alpha_2-1}. \tag{42}$$

Equating (34) - (38) to zeros, one can obtain the ML estimators of the unknown parameters numerically using the Newton-Raphson method, to obtain the ML estimates $\left(\hat{\alpha}_1,\hat{\lambda}_1,\hat{\alpha}_2,\hat{\lambda}_2 \; and \; \hat{p}\right)$. Also, the ML estimators of the rf and the hrf are obtained using the invariance property of the ML estimators by replacing the parameters in (19) and (20), respectively, by their ML estimators as follows

$$\hat{S}_M(x)=1-\hat{p}\left[1-e^{-2\left(e^{\hat{\lambda}_1 x}-1\right)}\right]^{\hat{\alpha}_1}-(1-\hat{p})\left[1-e^{-2\left(e^{\hat{\lambda}_2 x}-1\right)}\right]^{\hat{\alpha}_2}, \tag{43}$$

$$\hat{h}_M(x)$$
$$= \frac{2\hat{p}\hat{\alpha}_1\hat{\lambda}_1 e^{\hat{\lambda}_1 x-2\left(e^{\hat{\lambda}_1 x}-1\right)}\left[1-e^{-2\left(e^{\hat{\lambda}_1 x}-1\right)}\right]^{\hat{\alpha}_1-1}+2(1-\hat{p})\hat{\alpha}_2\hat{\lambda}_2 e^{\hat{\lambda}_2 x-2\left(e^{\hat{\lambda}_2 x}-1\right)}\left[1-e^{-2\left(e^{\hat{\lambda}_2 x}-1\right)}\right]^{\hat{\alpha}_2-1}}{1-\hat{p}\left[1-e^{-2\left(e^{\hat{\lambda}_1 x}-1\right)}\right]^{\hat{\alpha}_1}-(1-\hat{p})\left[1-e^{-2\left(e^{\hat{\lambda}_2 x}-1\right)}\right]^{\hat{\alpha}_2}}. \quad (44)$$

4.1.2 The asymptotic confidence intervals

In this subsection, an approximate method to construct *confidence intervals* (CIs) for the parameters based on asymptotic normality of the ML estimators is presented. Thus, the asymptotic distribution of the ML estimators can be used to obtain the CIs of the parameters of the MNTL-Ex distribution. The asymptotic variance-covariance matrix is obtained by the inverse of the asymptotic Fisher information matrix $I^{-1}(\underline{\theta})$.

Therefore, the two-sided approximate $100(1-\tau)\%$ CIs for the ML estimator say, $\hat{\theta}_i$ of a population value θ_i can be obtained by

$$p\left(-z \le \frac{\hat{\theta}_i - \theta_i}{\sqrt{v_{\hat{\theta}_i}}} \le z\right) = (1-\tau)$$

where Z is the $100(1-\tau)\%$. The two sided approximate$100(1-\tau)\%$ CIs for θ_i, will be given as follows:

$$L_{\theta_i} = \hat{\theta}_i - Z_{\frac{\tau}{2}}\sqrt{v_{\hat{\theta}_i}}, \qquad \text{and} \qquad U_{\theta_i} = \hat{\theta}_i + Z_{\frac{\tau}{2}}\sqrt{v_{\hat{\theta}_i}}, \quad i = 1,2,3,4,5.$$

where $\sqrt{v_{\hat{\theta}_i}}$ is the standard deviation.

4.2 Bayesian estimation

In this subsection, the unknown vector of parameters, $\underline{\theta} = (\alpha_1, \alpha_2, \lambda_1, \lambda_2, p)$, rf and hrf of the MNTL-Ex distribution, are estimated using the Bayesian estimation based on Type-II censored samples using the SE and LINEX loss functions.

Assuming that the prior belief of the experimenter is $p \sim Beta(a_1, a_2)$, $\alpha_1 \sim Gamma(b_1, b_2)$, $\alpha_2 \sim Gamma(b_3, b_4)$, $\lambda_1 \sim Gamma(b_5, b_6)$, and $\lambda_2 \sim Gamma(b_7, b_8)$.

Suppose that the parameters are independent, then the joint prior density function for $\underline{\theta} = (p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is given by

$$\pi(\underline{\theta}) = \pi_1(\alpha_1) \times \pi_2(\alpha_2) \times \pi_3(\lambda_1) \times \pi_4(\lambda_2) \times \pi_5(p),$$

$$\pi(\underline{\theta}) = \frac{1}{\mathbf{B}(a_1, a_2)} p^{a_1-1}(1-p)^{a_2-1} \frac{{b_2}^{b_1}{b_4}^{b_3}{b_6}^{b_5}{b_8}^{b_7}}{\Gamma(b_1)\Gamma(b_3)\Gamma(b_5)\Gamma(b_7)}$$
$$\times {\alpha_1}^{b_1-1}{\alpha_2}^{b_3-1}{\lambda_1}^{b_5-1}{\lambda_2}^{b_7-1} \times e^{-(\alpha_1 b_2+\alpha_2 b_4+\lambda_1 b_6+\lambda_2 b_8)}, \quad (45)$$

where a_i and b_j are the hyper parameters of the prior distribution, $0 < p < 1$, $\alpha_1, \alpha_2, \lambda_1, \lambda_2, a_i, b_j > 0$ and $a_i = a_1, a_2,\ b_j = b_1, b_2, \dots, b_8$.

Then, the joint posterior density function of $\underline{\theta} = (p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ can be obtained through multiplying (32) by (45) as given below

$$\pi(\underline{\theta}|\underline{x}) = k \times \varphi(\underline{\theta}, a_i, b_j) \times \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} \right] \right\} \times \left(1 - p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r}, \quad (46)$$

where

$$\varphi(\underline{\theta}, a_i, b_1) = \frac{1}{\mathbf{B}(a_1, a_2)} p^{a_1 - 1}(1-p)^{a_2 - 1} \frac{{b_2}^{b_1}{b_4}^{b_3}{b_6}^{b_5}{b_8}^{b_7}}{\Gamma b_1 \Gamma b_3 \Gamma b_5 \Gamma b_7} {\alpha_1}^{b_1 - 1}{\alpha_2}^{b_3 - 1}{\lambda_1}^{b_5 - 1}{\lambda_2}^{b_7 - 1} \times e^{-(\alpha_1 b_2 + \alpha_2 b_4 + \lambda_1 b_6 + \lambda_2 b_8)},$$

and k is a normalizing constant,

where $k^{-1} = \int_{\underline{\theta}} \pi(\underline{\theta}) L(\underline{\theta}|\underline{x}) d\underline{\theta}$

$$= \int_{\underline{\theta}} \omega(\underline{\theta}, a_i, b_j) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} \right] \right\} \times \left(1 - p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} d\underline{\theta},$$

where $\int_{\underline{\theta}} = \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty$ and $d\underline{\theta} = dp d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2$.

Thus, the marginal posterior distribution of each parameter can be obtained by integrating the joint posterior distribution (46) with respect to other parameters as follows, respectively,

The marginal posterior density of p is

$$\pi(p|\underline{x}) = k \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\underline{\theta}, a_i, b_j) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} \right] \right\} \times \left(1 - p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2,$$

$$0 < p < 1. \quad (47)$$

The marginal posterior density of α_1 is

$$\pi(\alpha_1|\underline{x}) = k\int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$
$$\left.\left.+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$
$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_2 d\lambda_1 d\lambda_{2,}$$
$$\alpha_1 > 0. \quad (48)$$

The marginal posterior density of α_2 is

$$\pi(\alpha_2|\underline{x}) = k\int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$
$$\left.\left.+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$
$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\lambda_1 d\lambda_{2,}$$
$$\alpha_2 > 0. \quad (49)$$

The marginal posterior density of λ_1 is

$$\pi(\lambda_1|\underline{x}) = k\int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$
$$\left.\left.+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$
$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\alpha_2 d\lambda_{2,}$$
$$\lambda_1 > 0. \quad (50)$$

Finally, the marginal posterior density of λ_2 is

$$\pi(\lambda_2|\underline{x}) = k\int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$
$$\left.\left.+2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$
$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\alpha_2 d\lambda_1,$$
$$\lambda_2 > 0. \quad (51)$$

4.2.1 Point estimation

In this subsection, the point estimators of the parameters, rf and hrf of MNTL-Ex distribution based on Type II censored samples under SE and LINEX loss functions.

I. Bayes estimators of the mixture of new Topp-Leone exponential distribution under the squared error loss function

The Bayes estimators based on the SE loss function is the mean of the posterior distribution, and can be derived as follows

$$\theta^{\star}_{j(SE)} = E\left(\theta_j \middle| \underline{x}\right) = \int_{\theta_j} \theta_j \pi\left(\underline{\theta}\middle|\underline{x}\right) d\theta_j\,, \qquad j = 1,2,3,4,5, \tag{52}$$

where, $\theta_1 = p, \theta_2 = \alpha_1, \theta_3 = \alpha_2, \theta_4 = \lambda_1, \theta_5 = \lambda_2$.

Thus, the Bayes estimators of the mixing proportion p is given by

$$p^{\star}_{(SE)} = E\left(p\middle| \underline{x}\right) = \int_0^1 p\, \pi\left(p\middle|\underline{x}\right) dp$$

$$= k \int_0^1 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty p\varphi\left(\underline{\theta}, a_i, b_j\right) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2\left(e^{\lambda_1 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1 - 1} \right.\right.$$
$$\left.\left. + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2\left(e^{\lambda_2 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2 - 1} \right] \right\}$$
$$\times \left(1 - p\left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2}\right)^{n-r} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 dp. \tag{53}$$

Similarly, the Bayes estimators of the parameters $\alpha_1, \alpha_2, \lambda_1$ *and* λ_2 are given, respectively, by

$$\alpha_{1}{}^{\star}_{(SE)} = E\left(\alpha_1\middle| \underline{x}\right) = \int_0^\infty \alpha_1 \pi\left(\alpha_1\middle|\underline{x}\right) d\alpha_1$$

$$= k \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \alpha_1\varphi\left(\underline{\theta}, a_i, b_j\right) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2\left(e^{\lambda_1 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1 - 1} \right.\right.$$
$$\left.\left. + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2\left(e^{\lambda_2 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2 - 1} \right] \right\}$$
$$\times \left(1 - p\left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_2 d\lambda_1 d\lambda_2 d\alpha_1, \tag{54}$$

$$\alpha_{2}{}^{\star}_{(SE)} = E\left(\alpha_2\middle| \underline{x}\right) = \int_0^\infty \alpha_2 \pi\left(\alpha_2\middle|\underline{x}\right) d\alpha_2$$

$$= k \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \alpha_2\varphi\left(\underline{\theta}, a_i, b_j\right) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2\left(e^{\lambda_1 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1 - 1} \right.\right.$$
$$\left.\left. + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2\left(e^{\lambda_2 x}-1\right)} \left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2 - 1} \right] \right\}$$
$$\times \left(1 - p\left[1 - e^{-2\left(e^{\lambda_1 x}-1\right)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2\left(e^{\lambda_2 x}-1\right)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\lambda_1 d\lambda_2 d\alpha_2, \tag{55}$$

$$\lambda_{1}{}^{\star}_{(SE)} = E\left(\lambda_1\middle| \underline{x}\right) = \int_0^\infty \lambda_1 \pi\left(\lambda_1\middle|\underline{x}\right) d\lambda_1$$

$$= k\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \lambda_1\varphi(\underline{\theta},a_i,b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$

$$\left.\left.+\,2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$

$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp\,d\alpha_1 d\alpha_2 d\lambda_2 d\lambda_1, \tag{56}$$

and

$$\lambda^\star_{2(SE)} = E(\lambda_2|\underline{x}) = \int_0^\infty \lambda_2\pi(\lambda_2|\underline{x})d\lambda_2$$

$$= k\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \lambda_2\varphi(\underline{\theta},a_i,b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x-2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1-1}\right.\right.$$

$$\left.\left.+\,2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x-2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2-1}\right]\right\}$$

$$\times\left(1-p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1}-(1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp\,d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2. \tag{57}$$

The Bayes estimators of the rf and hrf under SE loss function are the posterior expectations and can be obtained as follows:

$$S^*_{M(SE)}(x) = E(S_M(x)|\underline{x}) = \int_{\underline{\theta}} S_M(x)\pi(\underline{\theta}|\underline{x})d\underline{\theta}\ , \tag{58}$$

and

$$h^*_{M(SE)}(x) = E(h_M(x)|\underline{x}) = \int_{\underline{\theta}} h_M(x)\pi(\underline{\theta}|\underline{x})d\underline{\theta}\ , \tag{59}$$

where $S_M(x)$ and $h_M(x)$ are given in (19) and (20).

To obtain the Bayes estimates of the parameters, rf and hrf based on the SE loss function, (53) – (59) should be solved numerically utilizing the Metropolis - Hastings algorithm of *Markov chain Monte Carlo* (MCMC) method in R programming language.

II. Bayes estimators of the mixture of new Topp-Leone exponential distribution under the LINEX loss function

The Bayes estimators of the unknown parameters using the LINEX loss function can be obtained as follows:

$$\theta^\star_{(LINEX)} = -\frac{1}{a}\ln E\left(e^{-a\underline{\theta}}|\underline{x}\right), \qquad a \neq 0,$$

where $E\left(e^{-a\underline{\theta}}|\underline{x}\right) =$

$$\int_{\underline{\theta}} e^{-a\underline{\theta}} \pi(\underline{\theta}|\underline{x}) d\underline{\theta}. \qquad (60)$$

Hence, the Bayes estimators of the parameters $p, \alpha_1,\ \alpha_2, \lambda_1\ and\ \lambda_2$ of the MNTL-Ex distribution under the LINEX loss function are given, respectively, by

$$p^{\star}_{(LINEX)} = -\frac{1}{a}\ln E(e^{-ap}|\underline{x}) = -\frac{1}{a} ln \int_0^1 e^{-ap}\pi(p|\underline{x})dp$$

$$= -\frac{k}{a} ln \int_0^1 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-ap}\varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1}\right.\right.$$
$$\left.\left.+ 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1}\right]\right\}$$
$$\times\left(1 - p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2 dp, \qquad (61)$$

$$\alpha_{1(LINEX)}^{\star} = -\frac{1}{a}\ln E(e^{-a\alpha_1}|\underline{x}) = -\frac{1}{a} ln \int_0^{\infty} e^{-a\alpha_1}\pi(\alpha_1|\underline{x})d\alpha_1$$

$$= -\frac{k}{a} ln \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{1} e^{-a\alpha_1}\varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1}\right.\right.$$
$$\left.\left.+ 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1}\right]\right\}$$
$$\times\left(1 - p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_2 d\lambda_1 d\lambda_2 d\alpha_1, \qquad (62)$$

$$\alpha_{2(LINEX)}^{\star} = -\frac{1}{a}\ln E(e^{-a\alpha_2}|\underline{x}) = -\frac{1}{a} ln \int_0^{\infty} e^{-a\alpha_2}\pi(\alpha_2|\underline{x})d\alpha_2$$

$$= -\frac{k}{a} ln \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{1} e^{-a\alpha_2}\varphi(\underline{\theta}, a_i, b_j)\left\{\prod_{i=1}^{r}\left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)}\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1}\right.\right.$$
$$\left.\left.+ 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)}\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1}\right]\right\}$$
$$\times\left(1 - p\left[1-e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1-e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\lambda_1 d\lambda_2 d\alpha_2, \qquad (63)$$

$$\lambda_{1(LINEX)}^{\star} = -\frac{1}{a}\ln E(e^{-a\lambda_1}|\underline{x}) = -\frac{1}{a} ln \int_0^{\infty} e^{-a\lambda_1}\pi(\lambda_1|\underline{x})d\lambda_1$$

$$= -\frac{k}{a} ln \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 e^{-a\lambda_1} \varphi(\underline{\theta}, a_i, b_j) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} \right] \right\} \times \left(1 - p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\alpha_2 d\lambda_2 d\lambda_1, \tag{64}$$

and

$$\lambda^{\star}_{2(LINEX)} = -\frac{1}{a} \ln E\left(e^{-a\lambda_2} | \underline{x}\right) = -\frac{1}{a} ln \int_0^\infty e^{-a\lambda_2} \pi\left(\lambda_2 | \underline{x}\right) d\lambda_2$$

$$= -\frac{k}{a} ln \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 e^{-a\lambda_2} \varphi(\underline{\theta}, a_i, b_j) \left\{ \prod_{i=1}^{r} \left[2p\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x}-1)} \left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1 - 1} + 2(1-p)\alpha_2\lambda_2 e^{\lambda_2 x - 2(e^{\lambda_2 x}-1)} \left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2 - 1} \right] \right\} \times \left(1 - p\left[1 - e^{-2(e^{\lambda_1 x}-1)}\right]^{\alpha_1} - (1-p)\left[1 - e^{-2(e^{\lambda_2 x}-1)}\right]^{\alpha_2}\right)^{n-r} dp d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2, \tag{65}$$

The Bayes estimators of the rf and hrf under the LINEX loss function can be obtained as given below

$$S^{*}_{M(LINX)}(x) = -\frac{1}{a} ln\, E\left(e^{-aS_M(x)} | \underline{x}\right) = -\frac{1}{a}\; ln \int_{\underline{\theta}} e^{-aS_M(x)} \pi\left(\underline{\theta} | \underline{x}\right) d\underline{\theta}, \tag{66}$$

and

$$h^{*}_{M(LINX)}(x) = -\frac{1}{a} ln\, E\left(e^{-ah_M(x)} | \underline{x}\right) = -\frac{1}{a} ln \int_{\underline{\theta}} e^{-ah_M(x)} \pi\left(\underline{\theta} | \underline{x}\right) d\underline{\theta}\;, \tag{67}$$

where $S_M(x)$ and $h_M(x)$ are defined in (19) and (20).

To obtain the Bayes estimates of the parameters, rf and hrf based on the LINEX loss function, (61)-(67) can be solved numerically using the Metropolis - Hastings algorithm of MCMC method.

4.1.1 Credible intervals

The Bayesian counterpart of a confidence interval is called a credibility interval. Then, this section is devoted to derive the *credible intervals* (CIs) of the parameters of the MNTL-Ex distribution. In general, a two-sided, $\left(L_j(\underline{x}), U_j(\underline{x})\right)$, $100(1-\tau)\%$ CIs of $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) =$

$(p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ are given by:

$$P\left[L_j(\underline{x}) < \theta_j < U_j(\underline{x}) \middle| \underline{x}\right] = \int_{L_j(\underline{x})}^{U_j(\underline{x})} \pi\left(\theta_j \middle| \underline{x}\right) d\theta_j = 1 - \tau, \quad j = 1,2,3,4,5, \quad (68)$$

where

$L_j(\underline{x})$ and $U_j(\underline{x})$ are the *lower limit* (LL) and the *upper limit* (UL), and $(1-\tau)$ is the credible coefficient.

Since, the marginal posterior distributions of the parameters $\underline{\theta} = (p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is given by (47)-(51), then a two-sided $100(1-\tau)\%$ CIs of $\underline{\theta} = (p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$ can be given by:

$$P\left[\theta_j < L_j(\underline{x}) \middle| \underline{x}\right] = \int_{L_j(\underline{x})}^{\infty} \pi\left(\theta_j \middle| \underline{x}\right) d\theta_j = 1 - \frac{\tau}{2}, \quad j = 1,2,3,4,5, \quad (69)$$

and

$$P\left[\theta_j < U_j(\underline{x}) \middle| \underline{x}\right] = \int_{U_j(\underline{x})}^{\infty} \pi\left(\theta_j \middle| \underline{x}\right) d\theta_j = \frac{\tau}{2}, \quad j = 1,2,3,4,5. \quad (70)$$

To obtain the two-sided $100(1-\tau)\%$ CIs of $\underline{\theta} = (p, \alpha_1, \alpha_2, \lambda_1, \lambda_2)$, Equations (69) and (70) can be solved numerically.

5. Simulation Study

In this section, a simulation study is performed to determine the efficiency of the ML and Bayes estimators under the SE and LINEX loss functions for different samples from the MNTL-Ex $(p_i, \alpha_i, \lambda_i)$ distribution for different samples of size (n=50, 100 and 150). The number of failure times (r) is selected at the following censoring levels: 50%, 70% and 100% (complete samples). The Monte Carlo simulation study is used to illustrate the performance of the ML. All the results for the ML method were calculated using Mathematica 11.

The computations of the ML estimation are carried out using number of replications (NR) =1000 times. The averages, estimated risks (ERs), Biases of the ML averages of the parameters, rf and hrf are computed for each model parameters and for each sample size as follows:

Average = $\frac{\sum_{i=1}^{N}(estimate)}{NR}$,

ERs = $\frac{\sum_{i=1}^{N}(estimated\ value - true\ value)^2}{NR}$,

(Bias)2 = $(estimated\ value - true\ value)^2$.

Table 1 shows the ML averages, ERs, Biases of the ML estimates and 95% confidence intervals (CIs) of the unknown parameters p_i, α_i, λ_i, rf and hrf for different sample size where the population parameter values are (α1=0.8, λ1=1.1, p=0.4, α2=0.16, λ2=1.2) and t_0=0.8. Table 2 presents the same computational results for the same values of parameters with the mixing proportion p=0.2.

Markov Chain Monte Carlo Method

The *Markov Chain Monte Carlo* (MCMC) method is a powerful computational technique used to approximate the posterior distributions of model parameters when analytical solutions are intractable. In this study, the MCMC method is employed to estimate the parameters of the proposed mixture distribution under the Bayesian framework. Specifically, the Metropolis-Hastings algorithm is utilized, a popular MCMC method, to generate samples from the posterior distribution.

The algorithm iteratively constructs a Markov chain by proposing candidate values for the parameters and accepting or rejecting them based on their likelihood ratio and a random acceptance criterion. This ensures that the chain converges to the target posterior distribution after enough iterations. The convergence of the MCMC is assessed chains using diagnostic tools such as trace plots and the Gelman-Rubin statistic.

In our analysis, informative priors are assigned to the parameters to reflect minimal prior knowledge, allowing the data to primarily inform the posterior estimates. The MCMC algorithm is implemented with a burn-in period of 10000 sampling iterations to ensure accurate posterior summaries.

The MCMC is applied to obtain the Bayes estimates across different sample sizes, while the results for the Bayes estimates were evaluated using the R programming language.

- Tables 3 and 4 display the Bayes averages, ERs, *relative absolute biases* (RABs) and 95% CIs of the unknown parameters p_i, α_i, λ_i for different sample sizes under the SE and LINEX loss functions. In Table 3 the population parameter values are (α_1=0.8, λ_1=1.1, p=0.4, α_2=0.16, λ_2=1.2) and NR=10000, while in Table 4 the population parameter values are (α_1=0.8, λ_1=1.1, p=0.9, α_2=0.16, λ_2=1.2).

- Tables 5 and 6 present the Bayes averages, ERs, RABs and 95% CIs of the rf and hrf at t_0=0.8, with the same values for the population parameters used in Tables 3 and 4.

Table 1: ML averages of the estimates, estimated risks, biases and 95% confidence intervals of the parameters, rf and hrf from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 90% and replications NR=1000 (α_1=0.8, λ_1=1.1, p=0.4, α_2=0.16, λ_2=1.2)

n	r	Parameters	Averages	ER	Bias	UI	LI	Length
50	25	α_1	1.47178	0.45436	0.45129	1.58041	1.36315	0.21726
		λ_1	0.47307	0.39526	0.39305	0.56535	0.38078	0.18457
		p	0.76729	0.13563	0.13490	0.82014	0.71445	0.10569
		α_2	0.22339	0.00455	0.00402	0.26860	0.17819	0.09041
		λ_2	0.53420	0.44725	0.44329	0.65751	0.41089	0.24662
		$S_M(t_0)$	0.71662	0.26205	0.26185	074446	0.68878	0.05568
		$h_M(t_0)$	0.67711	3.32834	3.32071	0.84839	0.50584	0.34255
	35	α_1	1.49875	0.49073	0.48825	1.59633	1.40117	0.19516
		λ_1	0.56576	0.28569	0.28541	0.59889	0.53264	0.06625
		p	0.79990	0.16059	0.15992	0.85055	0.74925	0.10130
		α_2	0.17884	0.00124	0.00035	0.23726	0.12041	0.11684
		λ_2	0.63302	0.32228	0.32147	0.68898	0.57706	0.11192
		$S_M(t_0)$	0.69746	0.24291	0.24261	0.73155	0.66338	0.06817
		$h_M(t_0)$	0.88684	2.6021	2.60033	0.96935	0.80434	0.16501
	50	α_1	1.51442	0.51263	0.51039	1.60729	1.42155	0.18574
		λ_1	0.62369	0.22689	0.22687	0.63455	0.61284	0.02171
		p	0.81428	0.17228	0.17163	0.86449	0.76406	0.10044
		α_2	0.15975	0.00098	0.00000	0.22119	0.09831	0.12288
		λ_2	0.68240	0.26845	0.26791	0.72802	0.63678	0.09124
		$S_M(t_0)$	0.68513	0.23095	0.23061	0.72106	0.64919	0.07186
		$h_M(t_0)$	1.0256	2.17324	2.17198	1.0952	0.95605	0.13917

Table 1: Continued

n	r	Parameters	Averages	ER	Bias	UI	LI	Length
100	50	α_1	1.4755	0.45776	0.45630	1.55029	1.40072	0.14957
		λ_1	0.47948	0.38591	0.38504	0.53714	0.42183	0.11531
		p	0.76917	0.13664	0.13629	0.80581	0.73254	0.07327
		α_2	0.22224	0.00410	0.00387	0.252	0.19248	0.05952
		λ_2	0.54332	0.43281	0.43123	0.62122	0.46542	0.15579
		$S_M(t_0)$	0.71488	0.26018	0.26008	0.73476	0.69501	0.03975
		$h_M(t_0)$	0.68856	3.28237	3.27914	0.79993	0.57718	0.22275
	70	α_1	1.50175	0.49354	0.49245	1.56624	1.43726	0.12898
		λ_1	0.56788	0.28327	0.28315	0.58965	0.54612	0.04353
		p	0.80231	0.16214	0.16185	0.83569	0.76893	0.06676
		α_2	0.17799	0.00071	0.00032	0.21678	0.13921	0.07756
		λ_2	0.63564	0.31884	0.31849	0.67206	0.59922	0.07284

		$S_M(t_0)$	0.698352	0.24361	0.24349	0.72037	0.67634	0.04403
		$h_M(t_0)$	0.89100	2.58784	2.58693	0.95034	0.83166	0.11869
	100	α_1	1.51295	0.50937	0.50829	1.57735	1.44855	0.12879
		λ_1	0.62361	0.22696	0.22694	0.63131	0.61592	0.01539
		p	0.81265	0.17058	0.17028	0.84686	0.77844	0.06842
		α_2	0.16285	0.00046	0.00008	0.20469	0.12101	0.08369
		λ_2	0.68353	0.26698	0.26674	0.71405	0.65302	0.06103
		$S_M(t_0)$	0.68412	0.22979	0.22964	0.70814	0.66009	0.04805
		$h_M(t_0)$	1.02567	2.17249	2.17188	1.07406	0.97726	0.09679
150	75	α_1	1.47729	0.45962	0.45872	1.53587	1.41871	0.11716
		λ_1	0.48219	0.38219	0.38169	0.52617	0.43820	0.08797
		p	0.77052	0.1375	0.13728	0.79942	0.74161	0.05781
		α_2	0.22126	0.00392	0.00375	0.24626	0.19626	0.05000
		λ_2	0.54686	0.42753	0.42659	0.60665	0.48706	0.11959
		$S_M(t_0)$	0.71456	0.25982	0.25975	0.73095	0.69818	0.03277
		$h_M(t_0)$	0.69369	3.26255	3.26057	0.78085	0.60653	0.17432
	105	α_1	1.50101	0.49211	0.49142	1.55258	1.44944	0.10314
		λ_1	0.56785	0.28327	0.28319	0.58586	0.54983	0.03603
		p	0.80134	0.16128	0.16108	0.82898	0.7737	0.05528
		α_2	0.17957	0.00064	0.00038	0.21091	0.14824	0.06267
		λ_2	0.63630	0.31802	0.31776	0.66810	0.60449	0.06361
		$S_M(t_0)$	0.69781	0.24305	0.24296	0.71642	0.67920	0.03722
		$h_M(t_0)$	0.89084	2.58806	2.58744	0.93955	0.84214	0.09741
	150	α_1	1.51302	0.50913	0.50839	1.56615	1.45989	0.10626
		λ_1	0.62371	0.22686	0.22685	0.63001	0.61741	0.01259
		p	0.81195	0.16990	0.16970	0.83958	0.78432	0.05526
		α_2	0.16432	0.00030	0.00002	0.19737	0.13127	0.06610
		λ_2	0.68486	0.26552	0.26537	0.70912	0.66059	0.04853
		$S_M(t_0)$	0.68378	0.22941	0.22931	0.70301	0.66453	0.03848
		$h_M(t_0)$	1.0255	2.17281	2.17238	1.06636	0.98463	0.08172

Table 2: ML averages of the estimates, estimated risks, biases and 95% confidence intervals of the parameters, rf and hrf from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 100% and replications NR=1000

(α_1=0.8, λ_1=1.1, p=0.2, α_2=0.16, λ_2=1.2)

n	r	Parameters	Averages	ER	Bias	UI	LI	Length
50	25	α_1	1.5874	0.62066	0.62000	1.63781	1.53699	0.10082
		λ_1	0.49223	0.37018	0.36938	0.54763	0.43682	0.11081
		p	0.39772	0.03912	0.03909	0.40907	0.38637	0.02270
		α_2	0.26643	0.01144	0.01133	0.28723	0.24563	0.04160
		λ_2	0.59729	0.36463	0.36325	0.66996	0.52463	0.14533
		$S_M(t_0)$	0.50347	0.12919	0.12905	0.52668	0.48027	0.04641
		$h_M(t_0)$	0.48722	1.74962	1.74734	0.58081	0.39362	0.18719
	35	α_1	1.61302	0.66151	0.66099	1.65729	1.56875	0.08854
		λ_1	0.55747	0.29451	0.29433	0.58427	0.53068	0.05358
		p	0.40733	0.04303	0.04300	0.41776	0.39696	0.02079
		α_2	0.24179	0.00682	0.00669	0.26438	0.21920	0.04517
		λ_2	0.66366	0.28803	0.28765	0.70183	0.62549	0.07634
		$S_M(t_0)$	0.478211	0.11159	0.11154	0.49193	0.46449	0.02743
		$h_M(t_0)$	0.60915	1.44068	1.43985	0.66550	0.55279	0.11270
	50	α_1	1.62733	0.68500	0.68447	1.67268	1.58198	0.09069
		λ_1	0.60671	0.24336	0.24333	0.61703	0.59640	0.02062
		p	0.41253	0.04520	0.04517	0.42463	0.40043	0.02419
		α_2	0.22854	0.00483	0.00469	0.25141	0.20566	0.04575
		λ_2	0.70803	0.24219	0.24203	0.73289	0.68317	0.04972
		$S_M(t_0)$	0.46226	0.10116	0.10114	0.47102	0.45351	0.01751
		$h_M(t_0)$	0.70751	1.21377	1.21347	0.74148	0.67354	0.06794
100	50	α_1	1.59016	0.62463	0.62435	1.62295	1.55737	0.06558
		λ_1	0.49646	0.36458	0.36426	0.53178	0.46114	0.07064
		p	0.39866	0.03948	0.03947	0.40611	0.39122	0.01489
		α_2	0.26496	0.01107	0.01101	0.28014	0.24978	0.03035
		λ_2	0.60260	0.35743	0.35688	0.64859	0.55661	0.09199
		$S_M(t_0)$	0.50176	0.12788	0.12782	0.51683	0.48668	0.03015
		$h_M(t_0)$	0.49390	1.7307	1.72971	0.55551	0.43229	0.12322
	70	α_1	1.61370	0.66236	0.66211	1.64455	1.58286	0.06168
		λ_1	0.55865	0.29313	0.29305	0.57614	0.54117	0.03497
		p	0.40767	0.04314	0.04312	0.41515	0.40018	0.01497
		α_2	0.24141	0.00669	0.00662	0.25743	0.22538	0.03205
		λ_2	0.66525	0.28612	0.28595	0.69092	0.63958	0.05134
		$S_M(t_0)$	0.47782	0.11130	0.11128	0.48680	0.46884	0.01796
		$h_M(t_0)$	0.61133	1.43498	1.43462	0.64879	0.57386	0.07493

	Table 2: Continued							
	100	α_1	1.62750	0.68501	0.68476	1.65835	1.59666	0.06169
		λ_1	0.60722	0.24284	0.24283	0.61449	0.59995	0.01454
		p	0.41257	0.04520	0.04518	0.42066	0.40449	0.01617
		α_2	0.22789	0.00467	0.00461	0.24411	0.21166	0.03245
		λ_2	0.70889	0.24125	0.24118	0.72564	0.69215	0.03349
		$S_M(t_0)$	0.46182	0.10087	0.10086	0.46778	0.45585	0.01193
		$h_M(t_0)$	0.70896	1.21043	1.21028	0.73258	0.68533	0.04724
150	75	α_1	1.59023	0.62465	0.62446	1.61728	1.56318	0.05410
		λ_1	0.49638	0.36457	0.36435	0.52523	0.46753	0.05769
		p	0.39886	0.03955	0.03954	0.40491	0.39281	0.01209
		α_2	0.26475	0.01101	0.01097	0.27703	0.25247	0.02455
		λ_2	0.60238	0.35751	0.35714	0.64007	0.56469	0.07537
		$S_M(t_0)$	0.50179	0.12789	0.12785	0.51403	0.48955	0.02447
		$h_M(t_0)$	0.49375	1.73074	1.73009	0.54374	0.44377	0.09997
	105	α_1	1.61372	0.66232	0.66214	1.63963	1.58782	0.05181
		λ_1	0.55949	0.29219	0.29214	0.57401	0.54498	0.02903
		p	0.40782	0.04319	0.04319	0.41378	0.40185	0.01193
		α_2	0.24101	0.00661	0.00656	0.25436	0.22767	0.02669
		λ_2	0.66632	0.28492	0.28480	0.68719	0.64546	0.04172
		$S_M(t_0)$	0.47744	0.11104	0.11102	0.48498	0.46989	0.01508
		$h_M(t_0)$	0.61321	1.43036	1.43010	0.64479	0.58164	0.06315
	150	α_1	1.62751	0.68492	0.68476	1.65225	1.60276	0.04948
		λ_1	0.60726	0.24279	0.24278	0.61299	0.60154	0.01145
		p	0.41250	0.04516	0.04515	0.41901	0.40599	0.01302
		α_2	0.22811	0.00468	0.00463	0.24111	0.21511	0.02600
		λ_2	0.70914	0.24099	0.24094	0.72304	0.69524	0.02779
		$S_M(t_0)$	0.46183	0.10088	0.10087	0.46658	0.45709	0.00949
		$h_M(t_0)$	0.70886	1.21059	1.21050	0.72721	0.69050	0.03671

Table 3

Bayes averages of the estimates, estimated risks, RABs and 95% confidence intervals of the parameters, from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 100%, (α_1=0.8, λ_1=1.1, p=0.4, α_2=0.16, λ_2=1.2 and NR=10000)

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
50	25	SE	α_1	0.7865	0.0723	0.0168	0.8061	0.7664	0.0397
			λ_1	1.1221	0.1970	0.0201	1.1352	1.0990	0.0362
			p	0.3834	0.1091	0.0412	0.3976	0.3733	0.0243
			α_2	0.1842	0.2356	0.1517	0.2099	0.1648	0.0451
			λ_2	1.1986	0.0007	0.0011	1.2073	1.1832	0.0241
		LINEX	α_1	0.7938	0.0152	0.0077	0.8081	0.7758	0.0323
			λ_1	1.0795	0.1674	0.0185	1.1014	1.0528	0.0486
			p	0.4109	0.0477	0.0273	0.4242	0.3985	0.0257
			α_2	0.1668	0.0188	0.0428	0.1908	0.1500	0.0408
			λ_2	1.2105	0.0441	0.0087	1.2287	1.1978	0.0309
	35	SE	α_1	0.8100	0.0405	0.0125	0.8215	0.8003	0.0212
			λ_1	1.1130	0.0685	0.0119	1.1415	1.0919	0.0496
			p	0.4028	0.0033	0.0071	0.4140	0.3899	0.0241
			α_2	0.1651	0.0106	0.0322	0.1773	0.1540	0.0233
			λ_2	1.1973	0.0027	0.0021	1.2070	1.1862	0.0208
		LINEX	α_1	0.7999	1.80e-06	8.38e-05	0.8160	0.7863	0.0297
			λ_1	1.1007	2.00e-04	6.42e-04	1.1091	1.0919	0.0172
			p	0.3906	3.49e-02	2.33e-02	0.3982	0.3835	0.0147
			α_2	0.1617	1.16e-03	1.06e-02	0.1750	0.1515	0.0235
			λ_2	1.1827	1.18e-01	1.43e-02	1.2014	1.1661	0.0353
	50	SE	α_1	0.8095	3.64e-02	0.0119	0.8210	0.7951	0.0259
			λ_1	1.1131	6.96e-02	0.0120	1.1232	1.1038	0.0194
			p	0.4117	5.50e-02	0.0293	0.4260	0.3971	0.0289
			α_2	0.1892	3.42e-01	0.1829	0.2122	0.1639	0.0483
			λ_2	1.2030	3.61e-03	0.0025	1.2155	1.1943	0.0212
		LINEX	α_1	0.7997	1.86e-05	0.0002	0.8059	0.7927	0.0132
			λ_1	1.1055	1.21e-02	0.0050	1.1143	1.0996	0.0147
			p	0.4169	1.15e-01	0.0424	0.4284	0.4076	0.0208
			α_2	0.1607	2.31e-04	0.0047	0.1770	0.1392	0.0378
			λ_2	1.2029	3.49e-03	0.0024	1.2295	1.1835	0.0460

Table 3: Continued

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
100	50	SE	α_1	0.7920	0.0255	0.0099	0.8034	0.7750	0.0284
			λ_1	1.1022	0.0019	0.0020	1.1091	1.0942	0.0149
			p	0.3777	0.1981	0.0556	0.3973	0.3566	0.0407
			α_2	0.1688	0.0314	0.0553	0.1782	0.1555	0.0227
			λ_2	1.2008	0.0003	0.0007	1.2093	1.1891	0.0202
		LINEX	α_1	0.8108	0.0467	0.0135	0.8240	0.7907	0.0333
			λ_1	1.1156	0.0981	0.0142	1.1308	1.1049	0.0259
			p	0.3860	0.0773	0.0347	0.4015	0.3684	0.0331
			α_2	0.1501	0.0387	0.0614	0.1619	0.1383	0.0236
			λ_2	1.2014	0.0008	0.0012	1.2179	1.1891	0.0288
	70	SE	α_1	0.7991	0.0002	0.0010	0.8108	0.7879	0.0229
			λ_1	1.0957	0.0072	0.0038	1.1035	1.0900	0.0135
			p	0.4030	0.0036	0.0075	0.4102	0.3921	0.0181
			α_2	0.1578	0.0017	0.0131	0.1659	0.1487	0.0172
			λ_2	1.1911	0.0315	0.0074	1.2000	1.1790	0.0210
		LINEX	α_1	0.8092	0.0338	0.0115	0.8175	0.7989	0.0186
			λ_1	1.1037	0.0057	0.0034	1.1176	1.0931	0.0245
			p	0.4137	0.0755	0.0343	0.4287	0.3995	0.0292
			α_2	0.1267	0.4410	0.2075	0.1580	0.1041	0.0539
			λ_2	1.2030	0.0037	0.0074	1.2153	1.1919	0.0234
	100	SE	α_1	0.8056	0.0129	0.0071	0.8178	0.7937	0.0241
			λ_1	1.1132	0.0706	0.0120	1.1258	1.1008	0.0250
			p	0.3942	0.0132	0.0143	0.4020	0.3829	0.0191
			α_2	0.1608	0.0002	0.0052	0.1781	0.1497	0.0284
			λ_2	1.1968	0.0039	0.0026	1.2052	1.1879	0.0173
		LINEX	α_1	0.8058	0.0138	0.0073	0.8122	0.7975	0.0147
			λ_1	1.0884	0.0535	0.0105	1.0973	1.0781	0.0192
			p	0.3840	0.1016	0.0398	0.4027	0.3666	0.0361
			α_2	0.1594	0.0001	0.0031	0.1716	0.1486	0.0230
			λ_2	1.1856	0.0824	0.0119	1.2023	1.1686	0.0337

Table 3: Continued

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
150	75	**SE**	α_1	0.8026	0.0027	0.0032	0.8117	0.7941	0.0176
			λ_1	1.1008	0.0003	0.0008	1.1083	1.0941	0.0142
			p	0.4040	0.0064	0.0100	0.4126	0.3971	0.0155
			α_2	0.1684	0.0284	0.0526	0.1888	0.1536	0.0352
			λ_2	1.2158	0.1002	0.0131	1.2264	1.1959	0.0305
		LINEX	α_1	0.7931	0.0185	0.0085	0.8031	0.7833	0.0198
			λ_1	1.0801	0.1572	0.0180	1.1019	1.0613	0.0406
			p	0.4127	0.0645	0.0317	0.4315	0.3942	0.0373
			α_2	0.1550	0.0096	0.0307	0.1633	0.1442	0.0191
			λ_2	1.1832	0.1118	0.0139	1.2004	1.1731	0.0273
	105	**SE**	α_1	0.7962	5.48e-03	4.62e-03	0.8078	0.7878	0.0200
			λ_1	1.0991	2.73e-04	7.51e-04	1.1085	1.0823	0.0262
			p	0.4007	2.09e-04	1.80e-03	0.4127	0.3872	0.0255
			α_2	0.1500	3.95e-02	6.21e-02	0.1630	0.1418	0.0212
			λ_2	1.1813	1.39e-01	1.55e-02	1.1987	1.1524	0.0463
		LINEX	α_1	0.7876	0.0605	0.0153	0.8033	0.7691	0.0342
			λ_1	1.1089	0.0323	0.0081	1.1177	1.0994	0.0183
			p	0.4074	0.0220	0.0185	0.4214	0.3972	0.0242
			α_2	0.1669	0.0193	0.0435	0.1752	0.1592	0.0160
			λ_2	1.1829	0.1161	0.0142	1.2017	1.1697	0.0320
	150	**SE**	α_1	0.8015	0.0009	0.0019	0.8161	0.7839	0.0322
			λ_1	1.0935	0.0168	0.0059	1.1028	1.0863	0.0165
			p	0.4034	0.0046	0.0085	0.4158	0.3905	0.0253
			α_2	0.1500	0.0394	0.0620	0.1620	0.1365	0.0255
			λ_2	1.2002	0.0000	0.0002	1.2103	1.1949	0.0154
		LINEX	α_1	0.7905	0.0357	0.0118	0.8053	0.7709	0.0344
			λ_1	1.0902	0.0376	0.0088	1.1088	1.0751	0.0337
			p	0.4151	0.0923	0.0379	0.4312	0.4030	0.0282
			α_2	0.1395	0.1679	0.1280	0.1618	0.1222	0.0396
			λ_2	1.1981	0.0013	0.0015	1.2125	1.1772	0.0353

Table 4

Bayes averages of the estimates, estimated risks, RABs and 95% confidence intervals of the parameters, from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 100%, (α_1=0.8, λ_1=1.1, p=0.9, α_2=0.16, λ_2=1.2 and NR=10000)

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
50	25	SE	α_1	0.8234	0.2192	0.0292	0.8440	0.7995	0.0445
			λ_1	1.0760	0.2295	0.0217	1.0994	1.0628	0.0366
			p	0.8934	0.0173	0.0073	0.9004	0.8851	0.0153
			α_2	0.1497	0.0422	0.0642	0.1624	0.1339	0.0285
			λ_2	1.1963	0.0054	0.0030	1.2054	1.1875	0.0179
		LINEX	α_1	0.8062	0.0155	0.0077	0.8145	0.7997	0.0148
			λ_1	1.0864	0.0731	0.0122	1.1008	1.0723	0.0285
			p	0.8971	0.0032	0.0031	0.9044	0.8883	0.0161
			α_2	0.1531	0.0188	0.0428	0.1603	0.1458	0.0145
			λ_2	1.2151	0.0918	0.0126	1.2330	1.1930	0.0400
	35	SE	α_1	0.8003	4.54e-05	0.0004	0.8079	0.7897	0.0181
			λ_1	1.1185	1.38e-01	0.0169	1.1287	1.1030	0.0256
			p	0.8820	1.28e-01	0.0199	0.9001	0.8649	0.0351
			α_2	0.1535	1.66e-02	0.0403	0.1626	0.1465	0.0161
			λ_2	1.1978	1.86e-03	0.0017	1.2034	1.1904	0.0130
		LINEX	α_1	0.7897	0.0423	0.0128	0.7995	0.7790	0.0205
			λ_1	1.0955	0.0078	0.0040	1.1078	1.0814	0.0264
			p	0.8960	0.0061	0.0043	0.9017	0.8899	0.0118
			α_2	0.1522	0.0243	0.0487	0.1609	0.1427	0.0182
			λ_2	1.1909	0.0324	0.0075	1.2061	1.1772	0.0289
	50	SE	α_1	0.8095	3.65e-02	0.0119	0.8193	0.7972	0.0221
			λ_1	1.1079	2.55e-02	0.0072	1.1291	1.0936	0.0355
			p	0.9037	5.58e-03	0.0041	0.9107	0.8968	0.0139
			α_2	0.1597	2.05e-05	0.0014	0.1719	0.1514	0.0205
			λ_2	1.1971	3.24e-03	0.0023	1.2043	1.1888	0.0155
		LINEX	α_1	0.7900	0.0393	1.23e-02	0.8070	0.7666	0.0404
			λ_1	1.1129	0.0673	1.17e-02	1.1282	1.0998	0.0284
			p	0.9217	0.1892	2.41e-02	0.9446	0.8984	0.0462
			α_2	0.1727	0.0647	7.95e-02	0.1870	0.1562	0.0308
			λ_2	1.2100	0.0405	8.38e-03	1.2194	1.2014	0.0180

Table 4: Continued

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
100	50	SE	α_1	0.8230	2.12e-01	0.0288	0.8365	0.7985	0.0380
			λ_1	1.1067	1.83e-02	0.0061	1.1155	1.0942	0.0213
			p	0.9003	3.89e-05	0.0003	0.9095	0.8892	0.0203
			α_2	0.1668	1.87e-02	0.0427	0.1809	0.1525	0.0284
			λ_2	1.1868	6.90e-02	0.0109	1.2031	1.1698	0.0333
		LINEX	α_1	0.8286	0.3283	0.0358	0.8467	0.8034	0.0433
			λ_1	1.0981	0.0013	0.0016	1.1105	1.0767	0.0338
			p	0.9093	0.0351	0.0104	0.9230	0.8958	0.0272
			α_2	0.1618	0.0013	0.0114	0.1694	0.1537	0.0157
			λ_2	1.1799	0.1605	0.01669	1.1938	1.1674	0.0264
	70	SE	α_1	0.7898	0.0410	0.0126	0.8063	0.7721	0.0342
			λ_1	1.0986	0.0006	0.0011	1.1090	1.0855	0.0235
			p	0.9015	0.0009	0.0017	0.9179	0.8901	0.0278
			α_2	0.1639	0.0063	0.0249	0.1708	0.1576	0.0132
			λ_2	1.1771	0.2095	0.0190	1.2011	1.1565	0.0446
		LINEX	α_1	0.7937	0.0154	0.0077	0.8003	0.7877	0.0125
			λ_1	1.1145	0.0841	0.0131	1.1249	1.1024	0.0224
			p	0.8780	0.1920	0.0243	0.8977	0.8567	0.0410
			α_2	0.1757	0.0998	0.0987	0.1899	0.1588	0.0311
			λ_2	1.2229	0.2109	0.0191	1.2431	1.1996	0.0434
	100	SE	α_1	0.8053	0.0114	0.0066	0.8158	0.7959	0.0199
			λ_1	1.1260	0.2710	0.0236	1.1573	1.0976	0.0597
			p	0.9130	0.0680	0.0144	0.9264	0.8958	0.0306
			α_2	0.1764	0.1082	0.1028	0.1969	0.1582	0.0387
			λ_2	1.1860	0.0782	0.0116	1.1997	1.1736	0.0261
		LINEX	α_1	0.8078	0.0247	0.0098	0.8257	0.7976	0.0281
			λ_1	1.0940	0.0143	0.0054	1.1081	1.0814	0.0267
			p	0.8939	0.0146	0.0067	0.9048	0.8782	0.0266
			α_2	0.1441	0.0999	0.0988	0.1598	0.1279	0.0319
			λ_2	1.2051	0.0104	0.0042	1.2138	1.1976	0.0162

Table 4: Continued

n	r	Loss functions	Parameters	Averages	ER	RAB	UI	LI	Length
150	75	SE	α_1	0.8086	0.0299	0.0108	0.8214	0.7988	0.0226
			λ_1	1.1286	0.3289	0.0260	1.1482	1.1098	0.0384
			p	0.8943	0.0127	0.0062	0.9005	0.8884	0.0121
			α_2	0.1633	0.0043	0.0207	0.1774	0.1536	0.0238
			λ_2	1.1925	0.0224	0.0062	1.2014	1.1834	0.0180
		LINEX	α_1	0.7625	0.5623	4.68e-02	0.7969	0.7399	0.0570
			λ_1	1.0886	0.0510	1.02e-02	1.0995	1.0726	0.0269
			p	0.9056	0.0126	6.23e-03	0.9227	0.8974	0.0253
			α_2	0.1395	0.1675	1.27e-01	0.1626	0.1187	0.0439
			λ_2	1.1957	0.0073	3.57e-03	1.2029	1.1869	0.0160
	105	SE	α_1	0.8169	0.1144	0.0211	0.8270	0.7999	0.0271
			λ_1	1.0980	0.0015	0.0017	1.1059	1.0918	0.0141
			p	0.8939	0.0148	0.0067	0.9021	0.8870	0.0151
			α_2	0.1614	0.0008	0.0091	0.1678	0.1531	0.0147
			λ_2	1.1915	0.0285	0.0070	1.2105	1.1777	0.0328
		LINEX	α_1	0.8123	0.0607	0.0154	0.8264	0.7964	0.0300
			λ_1	1.0933	0.0175	0.0060	1.1008	1.0852	0.0156
			p	0.9049	0.0096	0.0054	0.9187	0.8877	0.0310
			α_2	0.1572	0.0031	0.0174	0.1653	0.1489	0.0164
			λ_2	1.2174	0.1211	0.0145	1.2370	1.2003	0.0367
	150	SE	α_1	0.7951	0.0095	6.10e-03	0.8077	0.7842	0.0235
			λ_1	1.1036	0.0051	3.27e-03	1.1191	1.0910	0.0281
			p	0.8988	0.0005	1.25e-03	0.9120	0.8887	0.0233
			α_2	0.1593	0.0001	4.13e-03	0.1764	0.1482	0.0282
			λ_2	1.19793	0.0017	1.72e-03	1.2061	1.1900	0.0161
		LINEX	α_1	0.7998	1.58e-05	0.0002	0.8119	0.7840	0.0279
			λ_1	1.0968	3.90e-03	0.0028	1.1099	1.0869	0.0230
			p	0.8869	6.77e-02	0.0144	0.8979	0.8761	0.0218
			α_2	0.1810	1.78e-01	0.1318	0.1934	0.1685	0.0249
			λ_2	1.2080	2.56e-02	0.0066	1.2244	1.1912	0.0332

Table 5: Bayes averages of the estimates, estimated risks, RABs and 95% confidence intervals of rf and hrf, from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 100%, (α_1=0.8, λ_1=1.1, p=0.4, α_2=0.16, λ_2=1.2 and NR=10000)

n	r	Loss functions	rf and hrf	Averages	ER	RAB	UI	LI	Length
50	25	SE	$S_{M(SE)}(t_0)$	0.0250	0.0015	0.0852	0.0326	0.0158	0.0168
			$h_{M(SE)}(t_0)$	17.3447	0.0255	0.0004	17.3548	17.3352	0.0196
		LINNEX	$S_{M(LINEX)}(t_0)$	0.0243	0.0006	0.0571	0.0330	0.0158	0.0172
			$h_{M(LINEX)}(t_0)$	17.3406	0.0060	0.0002	17.3507	17.3304	0.0203
	35	SE	$S_{M(SE)}(t_0)$	0.0400	0.1158	0.7385	0.0530	0.0243	0.0287
			$h_{M(SE)}(t_0)$	17.3446	0.0249	0.0004	17.3541	17.3361	0.0180
		LINEX	$S_{M(LINEX)}(t_0)$	0.0284	1.16e-02	2.34e-01	0.0372	0.0220	0.0152
			$h_{M(LINEX)}(t_0)$	17.3323	8.09e-03	2.59e-04	17.3508	17.3185	0.0323
	50	SE	$S_{M(SE)}(t_0)$	0.0231	7.31e-06	0.0058	0.0352	0.0135	0.0217
			$h_{M(SE)}(t_0)$	17.3463	3.62e-02	0.0005	17.3730	17.3304	0.0426
		LINEX	$S_{M(LINEX)}(t_0)$	0.0556	4.26e-01	1.4163	0.0759	0.0341	0.0418
			$h_{M(LINEX)}(t_0)$	17.3490	6.00e-02	0.0007	17.3583	17.3379	0.0204
100	50	SE	$S_{M(SE)}(t_0)$	0.0367	0.0747	0.5930	0.0476	0.0218	0.0258
			$h_{M(SE)}(t_0)$	17.3397	0.0034	0.0001	17.3484	17.3305	0.0179
		LINEX	$S_{M(LINEX)}(t_0)$	0.0299	0.0189	0.2984	0.0513	0.0159	0.0354
			$h_{M(LINEX)}(t_0)$	17.3422	0.0120	0.0003	17.3489	17.3359	0.0130
	70	SE	$S_{M(SE)}(t_0)$	0.0043	0.1391	0.8094	0.0284	0.0000	0.0284
			$h_{M(SE)}(t_0)$	17.3403	0.0049	0.0002	17.3519	17.3294	0.0225
		LINEX	$S_{M(LINEX)}(t_0)$	0.0332	0.0415	0.4423	0.0433	0.0205	0.0228
			$h_{M(LINEX)}(t_0)$	17.3302	0.0173	0.0003	17.3413	17.3205	0.0208
	100	SE	$S_{M(SE)}(t_0)$	0.0137	0.0347	0.4041	0.0291	0.0000	0.0291
			$h_{M(SE)}(t_0)$	17.3490	0.0604	0.0007	17.3645	17.3389	0.0256
		LINEX	$S_{M(LINEX)}(t_0)$	0.0182	0.0092	0.2092	0.0261	0.0093	0.0168
			$h_{M(LINEX)}(t_0)$	17.3333	0.0049	0.0002	17.3513	17.3166	0.0347

Table 5: Continued

n	r	Loss functions	rf and hrf	Averages	ER	RAB	UI	LI	Length
150	75	**SE**	$S_{M(SE)}(t_0)$	0.0223	0.0001	0.0300	0.0311	0.0091	0.0220
			$h_{M(SE)}(t_0)$	17.3346	0.0018	0.0001	17.3464	17.3278	0.0186
		LINEX	$S_{M(LINEX)}(t_0)$	0.0239	0.0003	0.0376	0.0341	0.0134	0.0207
			$h_{M(LINEX)}(t_0)$	17.3327	0.0067	0.0002	17.3409	17.3225	0.0184
	105	**SE**	$S_{M(SE)}(t_0)$	0.0058	1.18e-01	7.46e-01	0.0284	0.0000	0.0284
			$h_{M(SE)}(t_0)$	17.3364	5.70e-05	2.17e-05	17.3481	17.3257	0.0224
		LINEX	$S_{M(LINEX)}(t_0)$	0.0284	0.0115	0.2331	0.0444	0.0105	0.0339
			$h_{M(LINEX)}(t_0)$	17.3312	0.0123	0.0003	17.3403	17.3213	0.0190
	150	**SE**	$S_{M(SE)}(t_0)$	0.0382	0.0920	0.6583	0.0592	0.0152	0.0440
			$h_{M(SE)}(t_0)$	17.3441	0.0217	0.0004	17.3577	17.3302	0.0275
		LINEX	$S_{M(LINEX)}(t_0)$	0.0197	0.0043	0.1431	0.0258	0.0128	0.0130
			$h_{M(LINEX)}(t_0)$	17.3201	0.1111	0.0009	17.3412	17.3057	0.0355

Table 6

Bayes averages of the estimates, estimated risks, RABs and 95% confidence intervals of rf and hrf, from the MNTL-Ex distribution for different sample size n, r=50%, 70% and 100%, (α_1=0.8, λ_1=1.1, p=0.9, α_2=0.16, λ_2=1.2 and NR=10000)

n	r	Loss functions	rf and hrf	Averages	ER	RAB	UI	LI	Length
50	25	**SE**	$S_{M(SE)}(t_0)$	0.0498	0.0148	0.1391	0.0619	0.0375	0.0244
			$h_{M(SE)}(t_0)$	15.7722	0.1045	0.0010	15.7840	15.7596	0.0244
		LINEX	$S_{M(LINEX)}(t_0)$	0.0460	0.0021	0.0527	0.0537	0.0381	0.0156
			$h_{M(LINEX)}(t_0)$	15.7840	0.3126	0.0017	15.7993	15.7588	0.0405
	35	**SE**	$S_{M(SE)}(t_0)$	0.0365	2.08e-02	0.1650	0.0448	0.0284	0.0164
			$h_{M(SE)}(t_0)$	15.7372	1.42e-01	0.0011	15.7492	15.7298	0.0194
		LINEX	$S_{M(LINEX)}(t_0)$	0.0487	0.0098	0.1135	0.0579	0.0377	0.0202
			$h_{M(LINEX)}(t_0)$	15.7532	0.0033	0.0001	15.7637	15.7418	0.0219
	50	**SE**	$S_{M(SE)}(t_0)$	0.0357	2.54e-02	0.1823	0.0520	0.0240	0.0280
			$h_{M(SE)}(t_0)$	15.7617	1.28e-02	0.0003	15.7714	15.7495	0.0219
		LINEX	$S_{M(LINEX)}(t_0)$	0.0533	0.0365	2.18e-01	0.0640	0.0410	0.0230
			$h_{M(LINEX)}(t_0)$	15.7574	0.0006	8.32e-05	15.7705	15.7467	0.0238

Table 6: Continued

n	r	Loss functions	rf and hrf	Averages	ER	RAB	UI	LI	Length
100	50	SE	$S_{M(SE)}(t_0)$	0.0441	6.32e-05	0.0090	0.0519	0.0337	0.0182
			$h_{M(SE)}(t_0)$	15.7488	2.12e-02	0.0004	15.7578	15.7417	0.0161
		LINEX	$S_{M(LINEX)}(t_0)$	0.0327	0.0487	0.2522	0.0425	0.0219	0.0206
			$h_{M(LINEX)}(t_0)$	15.7483	0.0240	0.0004	15.7570	15.7386	0.0184
	70	SE	$S_{M(SE)}(t_0)$	0.0581	0.0824	0.3281	0.0776	0.0338	0.0438
			$h_{M(SE)}(t_0)$	15.7802	0.2322	0.0015	15.7955	15.7548	0.0407
		LINEX	$S_{M(LINEX)}(t_0)$	0.0471	0.0046	0.0778	0.0525	0.0408	0.0117
			$h_{M(LINEX)}(t_0)$	15.7672	0.0492	0.0007	15.7758	15.7579	0.0179
	100	SE	$S_{M(SE)}(t_0)$	0.0669	0.2151	0.5301	0.0790	0.0475	0.0315
			$h_{M(SE)}(t_0)$	15.7646	0.0293	0.0005	15.7730	15.7538	0.0191
		LINEX	$S_{M(LINEX)}(t_0)$	0.0367	0.0193	0.1588	0.0488	0.0216	0.0272
			$h_{M(LINEX)}(t_0)$	15.7453	0.0462	0.0006	15.7557	15.7355	0.0202
150	75	SE	$S_{M(SE)}(t_0)$	0.0350	0.0305	0.1998	0.0455	0.0248	0.0207
			$h_{M(SE)}(t_0)$	15.7662	0.0407	0.0006	15.7766	15.7504	0.0262
		LINEX	$S_{M(LINEX)}(t_0)$	0.0391	0.0084	1.05e-01	0.0515	0.0257	0.0258
			$h_{M(LINEX)}(t_0)$	15.7547	0.0007	8.54e-05	15.7635	15.7480	0.0155
	105	SE	$S_{M(SE)}(t_0)$	0.0504	0.0177	0.1520	0.0749	0.0379	0.0370
			$h_{M(SE)}(t_0)$	15.7531	0.0034	0.0001	15.7606	15.7422	0.0184
		LINEX	$S_{M(LINEX)}(t_0)$	0.0397	0.0064	0.0916	0.0530	0.0265	0.0265
			$h_{M(LINEX)}(t_0)$	15.7829	0.2875	0.0017	15.7941	15.7666	0.0275
	150	SE	$S_{M(SE)}(t_0)$	0.04314	0.0001	1.37e-02	0.0510	0.0332	0.0178
			$h_{M(SE)}(t_0)$	15.7547	0.0007	8.73e-05	15.7662	15.7427	0.0235
		LINEX	$S_{M(LINEX)}(t_0)$	0.0343	3.49e-02	0.2137	0.0431	0.0244	0.0187
			$h_{M(LINEX)}(t_0)$	15.7498	1.54e-02	0.0003	15.7663	15.7323	0.0340

- From Tables 1 and 2, one can observe that when the sample size n and failure time (r) increase, the ERs and biases of the ML averages of the estimates for the parameters p_i, α_i, λ_i, rf and hrf decrease in almost cases. Moreover, the lengths of the CIs become narrower as the sample size increases.

- Tables 3 and 4 show that the ERs and RABs of the Bayes averages of the estimates decrease and the lengths of the CIs get shorter when the sample size n and failure times r increase in almost cases.

From Tables 5 and 6, one can notice that the ERs and RABs of the rf and hrf decrease and the lengths of the CIs get shorter when the sample size and failure times increases.

6. Applications

This section is devoted to illustrating the flexibility of the MNTL-Ex distribution in real-life applications using two real data sets. To validate the fitted model, the Kolmogorov-Smirnov (KS) goodness-of-fit test is performed on the data sets, where the p-value indicates that the model provides a good fit.

ML estimates of the parameters and the corresponding *standard errors* (Se), *Kolmogorov-Smirnov* (K-S) statistic and its corresponding p-value, the *log likelihood* (LL), *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC) and *consistent Akaike information criterion* (CAIC) are used to compare the fit of the competitor distributions, where

$AIC = 2m - 2\mathcal{L}, \qquad BIC = m\ln(n) - 2\mathcal{L}$

and

$$CAIC = AIC + 2\left(\frac{m(m+1)}{n-m-1}\right),$$

where $\mathcal{L}$ is the natural logarithm of the value of the likelihood function evaluated at the ML estimates, n is the number of the observations and m is the number of the estimated parameters. The best distribution corresponds to the lowest values of AIC, BIC and CAIC, also the highest p-value.

Data set I: The first real data set was presented by Baharith *et. al* (2021) which are the survival times in days of 72 guinea pigs infected with virulent tubercle bacilli. These data are:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, and 5.55.

Data set II: The second real data set refers to the survival times for 30 light bulbs in months. This data set given by Feroze and Aslam (2013). These data are:

0.0200, 0.1860, 0.2620, 0.6950, 0.9140, 1.9950, 0.0250, 0.1960, 0.3140, 0.7400, 0.9920, 2.2550, 0.0590, 0.1970, 0.5110, 0.7600, 1.1810, 2.5090, 0.0620, 0.2050, 0.6040, 0.8460, 1.1940, 2.9100, 0.1450, 0.2100, 0.6780, 0.8600, 1.3090 and 5.5430.

Table 7 presents the ML estimates of the parameters, rf, hrf and Ses from the MNTL-Ex distribution for the two real data sets. A comparison is provided between the proposed distribution and other fitted distributions, including one component of the MNTL-Ex distribution, the new Topp-Leone exponential (NTL-Ex) distribution from the proposed mixture in (17), and the mixture of two exponential (MEx) distributions by Jaheen (2005). The pdf of the NTL-Ex and MEx distributions are given by

$$f_{\text{NTL-Ex}}(x) = 2\alpha_1\lambda_1 e^{\lambda_1 x - 2(e^{\lambda_1 x} - 1)}\left[1 - e^{-2(e^{\lambda_1 x} - 1)}\right]^{\alpha_1 - 1},$$

and

$$f_{\text{MEx}}(x) = p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}.$$

The ML estimates of the unknown parameters, rf, hrf and the information criteria for four different distributions are given in Table 8. Table 9 gives the Bayes estimates, Ses, rf and hrf under the SE and LINEX loss functions for the two real data sets.

Figures 4-8 display the PP-plot, QQ-plot, empirical histogram, empirical scaled TTT-transform plot and box plot of the MNTL-Ex distribution for the first real data set, and Figures 9-13 exhibit the same plots for the second real data set. From these figures, it can be noted that the MNL-Ex distribution fits both real data sets well.

The TTT plot shows that the first real data set has increasing hrf, which suggests that guinea pigs infected with virulent tubercle bacilli are less likely to die early in their life cycle. However, as time passes, the mortality rate increases, indicating that the likelihood of death grows as the guinea pigs age and their health deteriorates over time.

The second data set has a decreasing hrf, which suggests that light bulbs are more likely to fail early in their lifespan due to initial defects or weaker components. As time progresses, the failure rate decreases, as bulbs that remain operational tend to be more robust and less likely to fail.

Table 7: ML estimates and standard errors, rf and

hrf of the MNTL-Ex distribution for the two real data sets

Parameters	**Data Set I**		**Data Set II**	
	Estimates	**Se**	**Estimates**	**Se**
α_1	3.8330	0.6939	8.1021	26.0322

λ_1	0.4012	0.0000	1.0462	0.4176
p	0.7866	0.1494	0.4942	0.0088
α_2	0.6754	0.0307	0.6777	0.0315
λ_2	1.1174	0.0003	0.8548	0.0600
rf	0.7252	0.1343	0.2758	0.0068
hrf	0.5430	0.5882	3.5500	5.0181

Table 8: ML estimates and information criteria

Parameters	**Data Set I**			**Data Set II**		
	MNTL-Ex	**NTL-Ex**	**MEx**	**MNTL-Ex**	**NTL-Ex**	**MEx**
α_1	3.8330	1.4404	__	8.1021	7.8581	__
λ_1	0.4012	0.2993	0.3504	1.0462	0.9527	0.1457
p	0.7866	__	0.5237	0.4942	__	0.3572
α_2	0.6754	__	__	0.6777	__	__
λ_2	1.1174	__	0.4832	0.8548	__	1.8731
rf	0.7252	0.7154	0.7192	0.2758	0.5693	0.4615
hrf	0.5430	0.6070	0.4102	3.5500	2.7469	0.6831
p-value	0.188	0.129	0.086	0.392	0.135	0.239
LL	-35.2712	225.847	238.642	-3082.92	1067.04	612.846
AIC	-25.2712	229.847	244.642	-3072.92	1071.04	618.846
BIC	-13.8879	234.4	251.472	-3065.92	1073.84	623.049
CAIC	-24.3621	230.021	244.995	-3070.42	1071.48	619.769

Table 8 shows that the MNTL-Ex distribution provides a significantly better fit than the other two distributions, as it has the highest p-value and the lowest values for the LL, AIC, BIC, and CAIC criteria for the two real data sets.

Table 9: Bayes estimates and standard errors, rf and

hrf from the MNTL-Ex distribution for the two real data sets

	Parameters	**Loss function**	
		Se	**LINEX**

		Bayes estimates	Standard errors	Bayes estimates	Standard errors
Data Set I	α_1	3.0040	0.0104	2.9918	0.0090
	λ_1	0.3895	0.0077	0.3894	0.0093
	p	0.3995	0.0080	0.4180	0.0114
	α_2	0.4863	0.0117	0.5114	0.0110
	λ_2	1.1096	0.0098	1.0964	0.0096
	rf	0.3568	0.0072	0.3593	0.0093
	hrf	1.3732	0.0078	1.3578	0.0086
Data Set II	α_1	4.0011	0.0121	3.9944	0.0171
	λ_1	0.2932	0.0157	0.3060	0.0129
	p	0.4106	0.0176	0.3867	0.0198
	α_2	2.0026	0.0116	1.9877	0.0129
	λ_2	1.1064	0.0138	1.0952	0.0126
	rf	0.4473	0.0164	0.4677	0.0151
	hrf	2.2175	0.0204	2.1798	0.0178

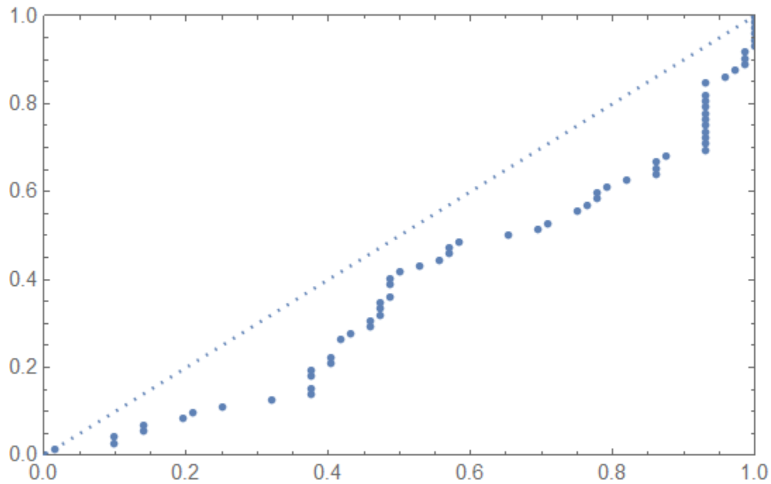

Figure 4. PP-plot of the MNTL-Ex distribution for the first real data set

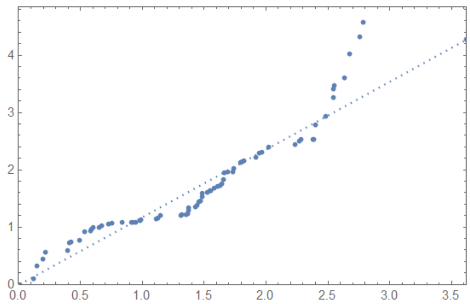

Figure 5. QQ-plot of the MNTL-Ex distribution for the first real data set

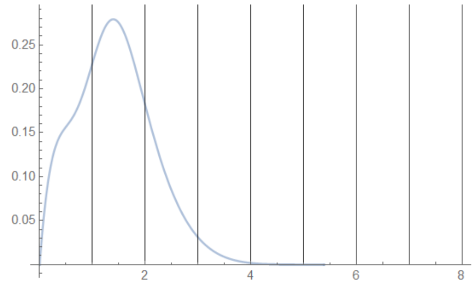

Figure 6. Empirical histogram plot for the first data set

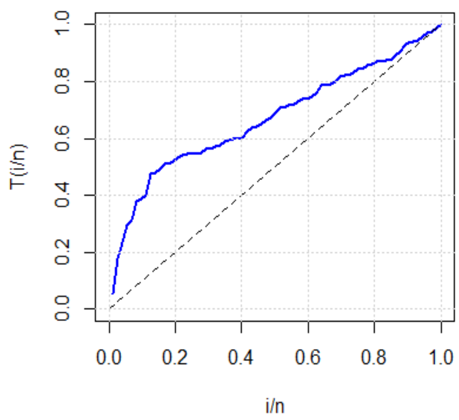

Figure 7. The empirical scaled TTT-transform plot for the first data set

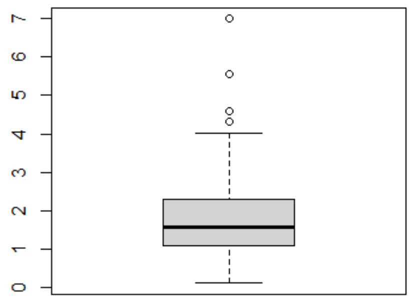

Figure 8. Box plot for the first data set

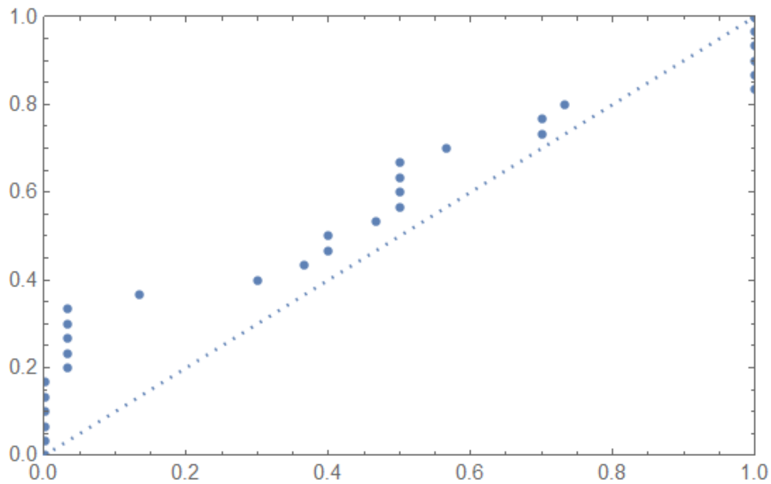

Figure 9. PP-plot of the MNTL-Ex distribution for the second real data set

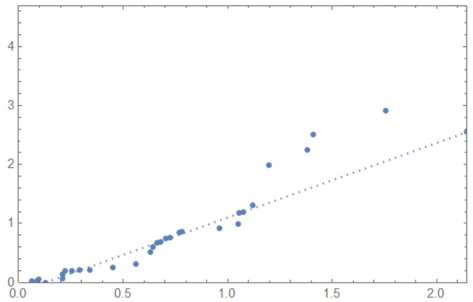

Figure 10. QQ-plot of the MNTL-Ex distribution for the second real data set

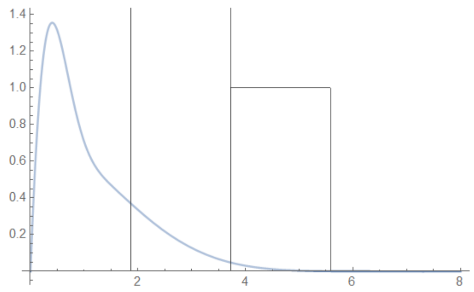

Figure 11. Empirical histogram plot for the second data set

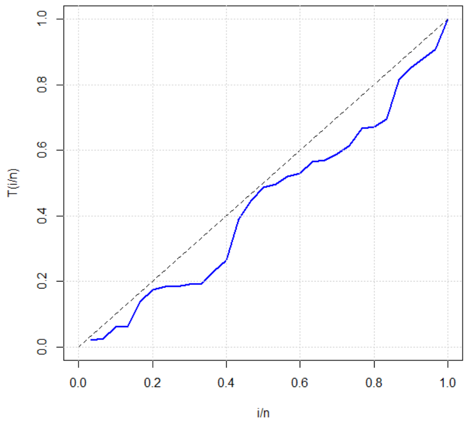

Figure 12. The empirical scaled TTT-transform plot for the second data set

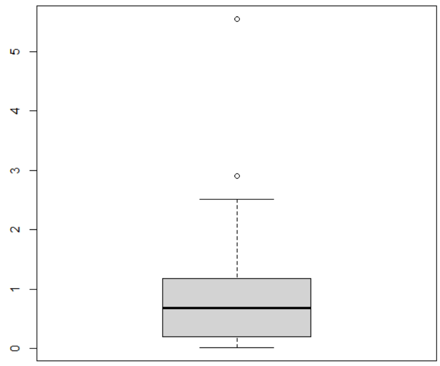

Figure 13. Box plot for the second data set

- One can observe the behavior shown in Figures 7 and 12, where one figure represents an increasing hazard rate function and the other a decreasing one, is indeed consistent with the flexibility of the proposed MNTL-Ex distribution. This behavior is a result of the model's ability to incorporate the characteristics of both increasing and decreasing hazard rates, depending on the values of the distribution's parameters.

7. Conclusion

In this paper, the MNTL-G family is introduced as a new family of continuous distributions. Some properties of this proposed mixture family are studied. The MNTL-Ex distribution is proposed as a sub-model from the mixture of two components NTL-G family. Certain properties, the ML and Bayes estimators are derived under the SE and LINEX loss functions for the unknown parameters of the MNTL-Ex distribution. A simulation study is conducted to assess the performance of the ML and Bayes estimates of the parameters of the MNTL-Ex distribution. Applicability and flexibility of the proposed distribution is illustrated by applying it to two real data sets. The results show that the values of information criteria are smaller for the MNTL-Ex as compared to the competing distributions, indicating that the proposed distribution is superior.

In future studies, the scope of this research could be extended to incorporate additional censoring schemes, as well as mixture distributions with more than two components. Both non-Bayesian and Bayesian prediction methods for the proposed distribution could also be explored. In future studies, we plan to investigate the performance of the proposed distribution when outliers are present, potentially introducing robust estimation techniques or modifications to the distribution to accommodate such data. Also, it may be useful to examine the impact of outliers on parameter estimation and the overall goodness-of-fit. Additionally, alternative estimation techniques, such as modified maximum likelihood, E-Bayesian, and empirical Bayesian estimation, may be applied.

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تقدير بييز والإمكان الأعظم للنماذج المختلطة لعائلة توب-ليون- الجديدة المعممة

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المستخلص:

تلعب التوزيعات المختلطة المحدودة دوراً هاماً في الكثير من التطبيقات الحياتية (الطبية، الهندسية، العلمية...وغيرها)، وذلك لأن التوزيعات المختلطة المحدودة تعتبر أسلوب إحصائي قوي لحل مشكلة البيانات غير المتجانسة التي ربما لا تتبع توزيعاً احتمالياً واحداً حيث أن كل جزء من مجتمع الدراسة قد يتبع توزيعاً معيناً أو قد يتبع جميع الأجزاء نفس التوزيع ولكن بمعالم مختلفة فيصبح المجتمع في هذه الحالة مزيج (خليط) من المجتمعات الجزئية وكل مجتمع له دالة كثافة احتمالية خاصة به.

الحافز الأساسي لإجراء هذه الدراسة يتمثل في تقديم خليط لاثنين من المكونات لعائلة توب ليون الجديدة المعممة كعائلة جديدة للتوزيعات المستمرة، والتي من خلالها يمكن الحصول على العديد من التوزيعات المختلطة الجديدة، كما تم دراسة بعض الخصائص الإحصائية لهذه العائلة المقترحة.

ففي هذه الدراسة على سبيل المثال تم اقتراح خليط مكونين من التوزيع الأسي - توب ليون- الجديدة المعممة كحالة خاصة من العائلة المقترحة. ومن ثم تم دراسة بعض الخصائص الإحصائية لهذا التوزيع الجديد، وكذلك تقدير المعالم ودالة الصلاحية ودالة الفشل باستخدام طريقة الإمكان الأعظم بناءً على العينات المراقبة من النوع الثاني وطريقة بييز اعتماداً على نوعين من دوال الخسارة وهما دالة خسارة مربع الخطأ ودالة الخسارة الأسية الخطية.

وفي النهاية، تم إجراء دراسة محاكاة لتقييم أداء مقدرات المعلمات ثم تطبيق وتحليل مجموعتين من البيانات الحقيقية لتأكيد نتائج المحاكاة.

الكلمات الإفتتاحية:

التوزيعات المختلطة، عائلة توب-ليون الجديدة ، التوزيع الأسي، تقدير الإمكان الأعظم، تقدير بييز.